UNIT-3

MATRICES

Definition : Matrix

A matrix is a rectangular array or arrangement of numbers in m rows and n columns, which are enclosed by a square brackets [ ] or paranthesis ( ).

The numbers that form a matrix are called <u>elements</u> of a matrix and matrix is represented as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m\times n}$$

Element of a matrix.

The numbers $a_{11}, a_{12}, \dots$ etc in the above matrix are known as elements of the matrix, generally represented as a_{ij}, which denotes the elements in i^{th} row and j^{th} column.

Order of a matrix.

The above matrix has m rows and n columns, then A is of order $m\times n$.

Types of matrices.

1. Null or Zero matrix :

In a matrix, if every elements are zero, it is known as zero matrix.

example, $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. Row matrix.

A matrix consisting of a single row is called a row matrix. The order of row matrix is $1\times n$.

example: $[1 \;\; 3 \;\; 2]_{1\times 3}$, $[2 \;\; -4]_{1\times 2}$

3. Column matrix :-

A matrix having single column is called a column matrix. The order of column matrix is $m \times 1$

example :- $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}_{3\times1}$ $\begin{bmatrix} 2 \\ -4 \end{bmatrix}_{2\times1}$

4. Rectangular matrix :-

A matrix in which the number of rows is not equal to number of columns is called a rectangular matrix

example : $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}_{2\times3}$

5. Square matrix :-

A matrix having same numbers of rows and columns is called a square matrix.

example : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$

Principal diagonal of a square matrix is the ordered set of elements a_{ij}, where $i=j$.

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 5 & 2 & 3 \\ 6 & 4 & 0 \end{bmatrix}$$

The principal diagonal elements are 1, 2 and 0.

6. Diagonal matrix :

A square matrix in which all elements are zero except those in the main or principal diagonal is called a diagonal matrix. Some elements of the principal diagonal may be zero but not all.

ex :- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

7. Scalar matrix :

A diagonal matrix in which all the diagonal elements are same.

example :- $B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

8. Identity matrix or Unit matrix.

A scalar matrix in which each diagonal element is 1 (unity)

example :- $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. Triangular matrix.

A square matrix is said to be triangular matrix, if all the elements above or below the principal diagonal are zero.

→ If all the elements above the principal diagonal are zero, the matrix is lower triangular.

ex:- $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 5 & 0 \\ 2 & 1 & 3 \end{bmatrix}$

→ If all the elements below the principal diagonal are zero, the matrix is upper triangular.

ex:- $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 3 \end{bmatrix}$

10. Sub Matrix :

The matrix formed by deleting of some rows or some columns from a matrix is called a submatrix.

ex:- $\begin{bmatrix} 2 & 6 & -3 & 4 \\ 1 & 5 & 7 & -2 \\ 1 & 0 & 5 & 2 \\ 8 & 9 & -6 & 3 \end{bmatrix}$ Sub Matrices are : $\begin{bmatrix} 2 & 6 & -3 \\ 1 & 5 & 7 \end{bmatrix}$ $\begin{bmatrix} -3 & 4 \\ 7 & -2 \\ 5 & 2 \end{bmatrix}$

11. Transpose of a matrix :-

A matrix obtained by interchanging its rows and columns is called Transpose of a matrix. Transpose of matrix is denoted by A' or A^T.

example :- $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 0 \end{bmatrix}_{2\times 3}$ then $A' = \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ -3 & 0 \end{bmatrix}_{3\times 2}$

NOTE :* If A is matrix of order $m\times n$, then A' is of order $n\times m$.

* $(A')' = A$.

12. Symmetric matrix.

A square matrix which is equal to its transpose is known as symmetric matrix.

i.e. $A = A'$

example :- $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 5 \\ 4 & 5 & 3 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 5 \\ 4 & 5 & 3 \end{bmatrix}$

13. Skew Symmetric matrix.

A square matrix is said to be skew symmetric if the transpose of the matrix equals its negative.

A square matrix A is called skew symmetrix, if $A' = -A$.

example :- $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$. $A' = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

$A' = -\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

$A' = -A$.

14. Equal matrices :- Two matrices A and B are said to be equal if and only if they have the same order and each element of matrix A is equal to

the corresponding element of matrix B i.e., for each i, j, $a_{ij} = b_{ij}$.

example 1 :- If $A = \begin{bmatrix} 2 & -1 \\ 8 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 8 & 0 \end{bmatrix}$ then $A = B$

example 2 :- $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4/2 & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$ then $A = B$

Operation on Matrices :-

1. Addition of matrices :-

Let A and B be two matrices of same order, then their sum A+B is a matrix whose elements are sum of the corresponding elements of A and B.

NOTE :- Two matrices are added or subtracted if and only if they have same order.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Then $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$

2. Subtraction of matrices :- (Difference of matrices)

Let A and B be two matrices of same order, then their difference A−B is a matrix whose elements are difference of the corresponding elements of A and B.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Then $A - B = A + (-B) = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{bmatrix}$

3. Scalar Multiplication.

Let A is a matrix and k is any scalar (real no.) then product kA is a scalar multiplication obtained by multiplying each element of A by k.

ex:- $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$

$6A = \begin{bmatrix} 12 & 18 & 30 \\ 6 & 12 & 18 \end{bmatrix}$, multiplying each element by 6

4. Multiplication of matrices (Matrix multiplication)

Two matrices A and B can be multiplied together to get the product matrix AB if, and only if, the number of columns of A is equal to the number of rows in B

ex:- let $A = \begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 3\times1+4\times3 & 3\times2+4\times4 \\ 5\times1+0\times3 & 5\times2+0\times4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3+12 & 6+16 \\ 5+0 & 10+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 15 & 22 \\ 5 & 10 \end{bmatrix}$$

∴ Product of matrices A and B = AB

$$= \begin{bmatrix} 1^{st}\text{ row of } A \times 1^{st}\text{ col. of } B & 1^{st}\text{ row of } A \times 2^{nd}\text{ col. of } B \\ 2^{nd}\text{ row of } A \times 1^{st}\text{ col. of } B & 2^{nd}\text{ row of } A \times 2^{nd}\text{ col. of } B \end{bmatrix}$$

Properties of Addition of matrices :-

1. Commutative property : $A+B = B+A$
2. Associative property : $(A+B)+C = A+(B+C)$
3. Existence of Additive identity : $A+O = O+A = A$
   [O - Zero matrix]
4. Existence of additive inverse :- $A+(-A) = O = (-A)+A$
5. Transpose property : $(A+B)^T = A^T + B^T$
6. Cancellation law :- $A+B = A+C$
   $\Rightarrow B = C$

Properties of Scalar multiplication of Matrix.

If A and B are two matrices then,

1. $K(A+B) = KA + KB$
2. $(K+C)A = KA + CA$
3. $(KA)' = KA'$

Properties of matrix multiplication.

1. Non-Commutative ie., $AB \neq BA$
2. $A(BC) = (AB)C$ (Associativity)
3. Distributive law :- $A(B+C) = AB + BC$
4. $(AB)' = B'A'$
5. $AI = IA = A$.

Worked Examples

Q1. If a matrix has 18 elements, what are the possible orders it can have? What if it has 5 elements.

Sol:- A matrix having 18 elements can have
$1\times18, 18\times1, 6\times3, 3\times6, 2\times9, 9\times2$ orders.
If A matrix has 5 elements then possible orders are $1\times5, 5\times1$.

Q2. Construct a 3×4 matrix, whose elements are given by $a_{ij} = 4i + j$.

Sol:- 3×4 is given by, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

$a_{11} = 4(1)+1 = 5$ $\quad a_{12} = 4(1)+2 = 6$ $\quad a_{13} = 4(1)+3 = 7$ $\quad a_{14} = 4(1)+4 = 8$

$a_{21} = 4(2)+1 = 9$ $\quad a_{22} = 4(2)+2 = 10$ $\quad a_{23} = 4(2)+3 = 11$ $\quad a_{24} = 4(2)+4 = 12$

$a_{31} = 4(3)+1 = 13$ $\quad a_{32} = 4(3)+2 = 14$ $\quad a_{33} = 4(3)+3 = 15$ $\quad a_{34} = 4(3)+4 = 16$

$$\therefore A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Q3. If $A = \begin{bmatrix} 7 & 2 \\ 8 & 6 \\ 9 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -7 \\ 3 & 1 \\ -8 & 5 \end{bmatrix}$. Find $A+B$ and $A-B$

Sol:- $$A+B = \begin{bmatrix} 7 & 2 \\ 8 & 6 \\ 9 & -6 \end{bmatrix} + \begin{bmatrix} 4 & -7 \\ 3 & 1 \\ -8 & 5 \end{bmatrix} = \begin{bmatrix} 7+4 & 2-7 \\ 8+3 & 6+1 \\ 9-8 & -6+5 \end{bmatrix}$$

$$\therefore A+B = \begin{bmatrix} 11 & -5 \\ 11 & 7 \\ 1 & -1 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 7 & 2 \\ 8 & 6 \\ 9 & -6 \end{bmatrix} - \begin{bmatrix} 4 & -7 \\ 3 & 1 \\ -8 & 5 \end{bmatrix} = \begin{bmatrix} 7-4 & 2-(-7) \\ 8-3 & 6-1 \\ 9-(-8) & -6-5 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 3 & 9 \\ 5 & 5 \\ 17 & -11 \end{bmatrix}$$

Q4. If $A = \begin{bmatrix} 1 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 2 \\ -3 & 4 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 5 \\ 7 & 8 & 3 \end{bmatrix}$

Find $A+B+C$ and $A-B+C$

Sol: $A+B+C = \begin{bmatrix} 1 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 2 \\ -3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 5 \\ 7 & 8 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+2 & 5-2+3 & 6+3+1 \\ 7+0+1 & 8+(-1)+4 & 9+2+5 \\ 10-3+7 & 11+4+8 & 12+5+3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 & 10 \\ 8 & 11 & 16 \\ 14 & 23 & 20 \end{bmatrix}$$

$A-B+C = \begin{bmatrix} 1 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 2 \\ -3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 5 \\ 7 & 8 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1-(4)+2 & 5-(-2)+3 & 6-3+1 \\ 7-0+1 & 8-(-1)+4 & 9-2+5 \\ 10-(-3)+7 & 11-4+8 & 12-5+3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10 & 4 \\ 8 & 13 & 12 \\ 20 & 15 & 10 \end{bmatrix}$$

Q5. If $A = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 \\ 7 & 6 \end{bmatrix}$, Find $5A-2B-2C$

Sol: $5A-2B-2C = 5\begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix} - 2\begin{bmatrix} -1 & 2 \\ 7 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 25 & 10 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 6 \\ 8 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 14 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 25-6-(-2) & 10-6-4 \\ 0-8-14 & 5-2-12 \end{bmatrix}$$

$$5A-2B-2C = \begin{bmatrix} 21 & 0 \\ -22 & -9 \end{bmatrix}$$

Q6. Solve for A and B, if $3A+2B=\begin{bmatrix}21 & 16 & 1\\ 21 & 2 & 12\end{bmatrix}$,

$2A-3B=\begin{bmatrix}-12 & -11 & 5\\ 1 & -16 & 8\end{bmatrix}$

Sol:

$$3A+2B=\begin{bmatrix}21 & 16 & 1\\ 21 & 2 & 12\end{bmatrix}\rightarrow(i)\times 2$$

$$2A-3B=\begin{bmatrix}-12 & -11 & 5\\ 1 & -16 & 8\end{bmatrix}\rightarrow(ii)\times 3$$

$$6A+4B=\begin{bmatrix}42 & 32 & 2\\ 42 & 4 & 24\end{bmatrix}\rightarrow(iii)$$

$$6A-9B=\begin{bmatrix}-36 & -33 & 15\\ 3 & -48 & 24\end{bmatrix}\rightarrow(iv)$$

(iii) − (iv)

$$6A+4B-\cancel{6A}+9B=\begin{bmatrix}42 & 32 & 2\\ 42 & 4 & 24\end{bmatrix}-\begin{bmatrix}-36 & -33 & 15\\ 3 & -48 & 24\end{bmatrix}$$

$$13B=\begin{bmatrix}42+36 & 32+33 & 2-15\\ 42-3 & 4+48 & 24-24\end{bmatrix}$$

$$13B=\begin{bmatrix}78 & 65 & -13\\ 39 & 52 & 0\end{bmatrix}$$

$$B=\frac{1}{13}\begin{bmatrix}78 & 65 & -13\\ 39 & 52 & 0\end{bmatrix}$$

$$B=\begin{bmatrix}6 & 5 & -1\\ 3 & 4 & 0\end{bmatrix}$$

Substitute B value in eqn (i)

$$3A + 2\begin{bmatrix} 6 & 5 & -1 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 21 & 16 & 1 \\ 21 & 2 & 12 \end{bmatrix}$$

$$3A + \begin{bmatrix} 12 & 10 & -2 \\ 6 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 21 & 16 & 1 \\ 21 & 2 & 12 \end{bmatrix}$$

$$3A = \begin{bmatrix} 21 & 16 & 1 \\ 21 & 2 & 12 \end{bmatrix} - \begin{bmatrix} 12 & 10 & -2 \\ 6 & 8 & 0 \end{bmatrix}$$

$$3A = \begin{bmatrix} 21-12 & 16-10 & 1-(-2) \\ 21-6 & 2-8 & 12-0 \end{bmatrix}$$

$$3A = \begin{bmatrix} 9 & 6 & 3 \\ 15 & -6 & 12 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} 9 & 6 & 3 \\ 15 & -6 & 12 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 5 & -2 & 4 \end{bmatrix}$$

HW

Q7. If $2A+B = \begin{bmatrix} 6 & 3 \\ 6 & -2 \end{bmatrix}$ and $3A+2B = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ find A and B.

Sol: $A = \begin{bmatrix} 11 & 6 \\ 12 & -9 \end{bmatrix}$ $B = \begin{bmatrix} -16 & -9 \\ -18 & 16 \end{bmatrix}$

Q8. If $A = \begin{bmatrix} 2 & 7 & 3 \\ 4 & -5 & 6 \end{bmatrix}$ Show that $(A')' = A$.

Sol:

$$A = \begin{bmatrix} 2 & 7 & 3 \\ 4 & -5 & 6 \end{bmatrix} \rightarrow (i)$$

$$A' = \begin{bmatrix} 2 & 4 \\ 7 & -5 \\ 3 & 6 \end{bmatrix}$$

$$(A')' = \begin{bmatrix} 2 & 7 & 3 \\ 4 & -5 & 6 \end{bmatrix} \rightarrow (ii)$$

(i) = (ii)

$\therefore$ $(A')' = A$.

Q 9. Compute the following $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

Sol:- $\begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Q.10. Compute the indicated product:

$$\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Sol:-

$$= \begin{bmatrix} 3\times2+(-1)\times1+3\times3 & 3\times(-3)+(-1)\times0+3\times1 \\ (-1)\times2+0\times1+2\times3 & (-1)\times(-3)+0\times0+2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-1+9 & -9+0+3 \\ -2+0+6 & 3+0+2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$$

Q11. Find the value of x, y and z.

(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

Sol:- (i) By using equality of matrices.

$y=4, x=1, z=3$

(ii) $x+y+z = 9 \rightarrow$ (i)

$x+z = 5 \rightarrow$ (ii)

$y+z = 7 \rightarrow$ (iii)

put $x+z=5$ in (i)

$y+5 = 9$

$y = 9-5$

$\boxed{y=4}$

put $y=4$ in (iii)

$4+z=7$

$z=7-4$

$\boxed{z=3}$

put $z=3$ in (ii)

$x+3=5$

$x=5-3$

$\boxed{x=2}$

Q12. If $A=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & -1\\ -3 & 2\end{bmatrix}$ find AB and BA. Is AB = BA.

Sol:- $AB = \begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ -3 & 2\end{bmatrix}$

$= \begin{bmatrix}2\times1+5\times(-3) & 2\times(-1)+5\times2\\ 1\times1+3\times(-3) & 1\times(-1)+3\times2\end{bmatrix}$

$= \begin{bmatrix}2-15 & -2+10\\ 1-9 & -1+6\end{bmatrix}$

$= \begin{bmatrix}-13 & +8\\ -8 & 5\end{bmatrix}$

$BA = \begin{bmatrix}1 & -1\\ -3 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$

$= \begin{bmatrix}1\times2+(-1)\times1 & 1\times5+(-1)\times3\\ -3\times2+2\times1 & -3\times5+2\times3\end{bmatrix}$

$= \begin{bmatrix}2-1 & 5-3\\ -6+2 & -15+6\end{bmatrix}$

$= \begin{bmatrix}1 & 2\\ -4 & -9\end{bmatrix}$

$AB \neq BA$.

Q13. If $A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$ Prove that $A^2-4A-5I=0$.

$$A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$$

$$A^2=A\times A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\times\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$$

$$=\begin{bmatrix}1\times1+2\times2+2\times2 & 1\times2+2\times1+2\times2 & 1\times2+2\times2+2\times1\\ 2\times1+1\times2+2\times2 & 2\times2+1\times1+2\times2 & 2\times2+1\times2+2\times1\\ 2\times1+2\times2+1\times2 & 2\times2+2\times1+1\times2 & 2\times2+2\times2+1\times1\end{bmatrix}$$

$$=\begin{bmatrix}1+4+4 & 2+2+4 & 2+4+2\\ 2+2+4 & 4+1+4 & 4+2+2\\ 2+4+2 & 4+2+2 & 4+4+1\end{bmatrix}$$

$$A^2=\begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}$$

$$4A=4\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}=\begin{bmatrix}4&8&8\\8&4&8\\8&8&4\end{bmatrix}$$

$$5I=\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$$

$$A^2-4A-5I=\begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}-\begin{bmatrix}4&8&8\\8&4&8\\8&8&4\end{bmatrix}-\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$$

$$=\begin{bmatrix}9-4-5 & 8-8-0 & 8-8-0\\ 8-8-0 & 9-4-5 & 8-8-0\\ 8-8-0 & 8-8-0 & 9-4-5\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$A^2-4A-5I=0$

Q14. Given $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$ find $(AB)'$ and $B'A'$

$$AB = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times2+2\times(-3)+(-1)\times(-1) & 4\times3+2\times0+(-1)\times5 \\ 3\times2+(-7)\times(-3)+(1)\times(-1) & 3\times3+(-7)\times0+1\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 8-6+1 & 12-5 \\ 6+21-1 & 9+5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 7 \\ 26 & 14 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 3 & 26 \\ 7 & 14 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 2 & -3 & -1 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & -7 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times4+(-3)\times2+(-1)\times(-1) & 2\times3+(-3)\times(-7)+(-1)\times1 \\ 3\times4+0\times2+5\times(-1) & 3\times3+0\times(-7)+5\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-6+1 & 6+21-1 \\ 12+0-5 & 9+5 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 3 & 26 \\ 7 & 14 \end{bmatrix}$$

Q15. The book shop of a particular school has 10 dozen chemistry books, 8 dozen of physics books, 10 dozen of economics books. Their selling prices are ₹80, ₹60 and ₹40 each respectively. Find the total amount the book shop will receive from selling all the books using matrix algebra..

Sol:- There are 10 dozen $= 10 \times 12 = 120$ chemistry books
8 dozen $= 8 \times 12 = 96$ physics books
10 dozen $= 10 \times 12 = 120$ economics books

The amount received by the book shop on selling books is product of books and price

$$= \begin{bmatrix} 120 & 96 & 120 \end{bmatrix} \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$$

$$= 120 \times 80 + 96 \times 60 + 120 \times 40$$

$$= 9600 + 5760 + 4800$$

$$= 20,160$$

$\therefore$ Amount received by selling books is ₹ 20,160.

Q16. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pay 5% interest per year and the second bond pays 7% interest per year. Using matrix multiplication determines how to divide ₹ 30,000 among the two types of bonds if the trust fund must obtain an annual total interest of ₹ 1800.

Sol:- Let the amount to be invested in first type bond be 'x' ₹ and second type bond is ₹ $(30000 - x)$.

$$\begin{bmatrix} x & 30,000 - x \end{bmatrix} \begin{bmatrix} 5/100 \\ 7/100 \end{bmatrix} = \begin{bmatrix} 1,800 \end{bmatrix}$$

$$\begin{bmatrix} \frac{5x}{100} + (30,000 - x)\frac{7}{100} \end{bmatrix} = \begin{bmatrix} 1,800 \end{bmatrix}$$

$$\Rightarrow \frac{5x}{100} + \frac{7(30000 - x)}{100} = 1800$$

$$\Rightarrow \frac{5x}{100} + \frac{210000}{100} - \frac{7x}{100} = 1800$$

$$\Rightarrow 2100 - \frac{2x}{100} = 1800$$

$$\Rightarrow 2100 - 1800 = \frac{2x}{100}$$

$$\Rightarrow 300 = \frac{x}{50}$$

$$\Rightarrow x = 300 \times 50$$

$$\Rightarrow \boxed{x = 15{,}000}$$

$$\therefore 30{,}000 - x = 15{,}000.$$

∴ The amount to be invested in first type bond is ₹15,000 and in second type of bond is ₹15,000

Determinant of a matrix.

The determinant of a matrix is a function that maps every square matrix to a unique number (real or complex). For any square matrix A, the determinant of A is denoted by det(A) or |A| or Δ (delta).

Determinant of 1×1 matrix

The determinant of 1×1 matrix $A = [a]$ is $|A| = a$.

Determinant of 2×2 matrix

The determinant of 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$|A| = ad - bc$$

Determinant of 3X3 matrix

The determinant of 3X3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is

$$|A| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$|A| = a(ei - hf) - b(di - fg) + c(dh - ge)$$

Worked Examples :-

1. Evaluate $\Delta = \begin{array}{ccc} (+) & (-) & (+) \\ \end{array}$
$$\Delta = \begin{vmatrix} 1 & 3 & 5 \\ -1 & 0 & 4 \\ 3 & 2 & 9 \end{vmatrix}$$

$$\Delta = 1(0-8) - 3(-9-12) + 5(-2-0)$$
$$= -8 - 3(-21) - 10$$
$$= -8 + 63 - 10$$
$$\Delta = 45.$$

2. Evaluate $\Delta = \begin{vmatrix} 0 & \cos\alpha & -\sin\alpha \\ -\cos\alpha & 0 & \sin\beta \\ \sin\alpha & \sin\beta & 0 \end{vmatrix}$

$$\Delta = 0\begin{vmatrix} 0 & \sin\beta \\ \sin\beta & 0 \end{vmatrix} - \cos\alpha \begin{vmatrix} -\cos\alpha & \sin\beta \\ \sin\alpha & 0 \end{vmatrix} - \sin\alpha \begin{vmatrix} -\cos\alpha \\ \sin\alpha \end{vmatrix}$$
$$= 0 - \cos\alpha(0 - \sin\alpha\sin\beta) - \sin\alpha(-\cos\alpha\sin\beta)$$
$$= \cos\alpha\sin\alpha\sin\beta + \cos\alpha\sin\alpha\sin\beta$$
$$\Delta = 2\cos\alpha\sin\alpha\sin\beta.$$

3. Evaluate $\Delta = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix}$ by expanding first column

$$\Delta = 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$
$$= 3(4-(-6)) - 0(8-3) + 1(-4-1)$$
$$= 3(10) - 0 + 1(-5)$$
$$= 30 - 5$$
$$\Delta = 25$$

4. Find the value of x, if $\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

Sol:- $\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$\Rightarrow 2 - 8 = 2x^2 - 24$

$\Rightarrow -6 = 2x^2 - 24$

$\Rightarrow 24 - 6 = 2x^2$

$\Rightarrow 18 = 2x^2$

$\Rightarrow x^2 = \frac{18}{2}$

$\Rightarrow x^2 = 9$

$\Rightarrow x = \pm\sqrt{9} = \pm 3$

5. Solve for x : $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$

Given, $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$

$\Rightarrow x\begin{vmatrix} 5 & x \\ 2 & x \end{vmatrix} - 2\begin{vmatrix} 2 & x \\ -1 & x \end{vmatrix} + (-1)\begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} = 0$

$\Rightarrow x(5x - 2x) - 2(2x - (-x)) - 1(4 - (-5)) = 0$

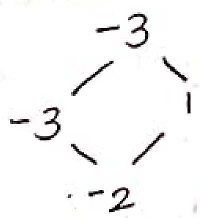
$\Rightarrow x(3x) - 2(2x + x) - 1(4 + 5) = 0$

$\Rightarrow 3x^2 - 6x - 9 = 0 \quad (\div 3)$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\therefore x = 3$ or $x = -1$



Properties of determinants.

Property 1:- Interchanging the corresponding rows and columns of a determinant does not change its value (i.e., $|A| = |A'|$)

Property 2:- If two rows or two columns of a determinant are interchanged, the sign of the determinant is changed.

Property 3:- If every element of a row or column of a determinant is zero, the value of the determinant is zero.

Property 4:- If two rows or columns of a determinant are identical, the value of the determinant is zero.

Property 5:- If every element of a row or column of a determinant is multiplied by the same constant k, the value of the determinant is multiplied by that constant.

Property 6:- The value of a determinant is not changed if each element of any row or of any column is added to (or subtracted) a constant multiple of the corresponding element of another row or column. $A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ B =

Property 7:- The determinant of a diagonal matrix is equal to the product of its diagonal elements.

Property 8:- The determinant of the product of two matrices is equal to the product of the determinants of the two matrices, that is $|AB| = |A||B|$

Property 9:- The determinant in which each element in any row, or column, consists of two terms, then the determinant can be expressed as the sum of two other determinants.

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

Minors, Cofactors and Adjoint of a matrix.

Minors :- Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

ex:- Find the minor of element 7 in the determinant

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Sol:- 7 Lies in 1 column & 3 row.

Its minor M_{31} is given by,

$$M_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12 - 15 = -3$$

Cofactor :- A cofactor is a number obtained when the minor M_{ij} of the element a_{ij} is multiplied by $(-1)^{i+j}$, where i and j represent the row and column of the particular element whose co-factor is being determined.

$$C_{ij} = M_{ij} \times (-1)^{i+j}$$

example:- $A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

$B = \begin{vmatrix} a_1+kc_1 & a_2+kc_2 & a_3+kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

WKT, $C_{ij} = (-1)^{i+j} M_{ij}$

Cofactor of 3 $= (-1)^{1+1}(8) = 8$

Cofactor of 4 $= (-1)^{1+2}(5) = -5$

Cofactor of 5 $= (-1)^{2+1}(4) = -4$

Cofactor of 8 $= (-1)^{2+2}(3) = 3$

Cofactor matrix $A = \begin{bmatrix} 8 & -5 \\ -4 & 3 \end{bmatrix}$

Adjoint of a matrix:-

The adjoint of a matrix A is the transpose of cofactor matrix. It is denoted by adj A.

ex:- Cofactor matrix $A = \begin{bmatrix} 8 & -5 \\ -4 & 3 \end{bmatrix}$

$$\text{Adj } A = \begin{bmatrix} 8 & -5 \\ -4 & 3 \end{bmatrix}^T = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

NOTE :-

* If A is a square matrix then $A \, \text{adj} A = \text{adj} A \cdot A = |A| I$ where I is a unit matrix of same order.
* If A and B are square matrix of same order then $|A.B| = |A|.|B|$.

Q. Find the minors and cofactors of the elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify that

$$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$$

Sol:- $a_{11} = 2$, $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix}$, $A_{11} = (-1)^{1+1} M_{11}$

$M_{11} = -20$ $\quad A_{11} = -20$

$a_{12} = -3$, $M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix}$, $A_{12} = (-1)^{1+2} M_{12}$

$= (-1)(-46)$

$= -42 - 4$

$M_{12} = -46$ $\quad A_{12} = 46$

$a_{13} = 5$, $M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$, $A_{13} = (-1)^{1+3} M_{13}$

$M_{13} = 30$ $\quad A_{13} = 30$

$a_{21} = 6$, $M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix}$, $A_{21} = (-1)^{2+1} M_{21}$

$= 21 - 25$ $\quad = (-1)(-4)$

$M_{21} = -4$ $\quad A_{21} = 4$

$a_{22} = 0$, $M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix}$, $A_{22} = (-1)^{2+2} M_{22}$

$= -14 - 5$ $\quad$ $A_{22} = -19$

$M_{22} = -19$

$a_{23} = 4$, $M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix}$, $A_{23} = (-1)^{2+3} M_{23}$

$= 10 + 3$ $\quad$ $= (-1)(13)$

$M_{23} = 13$ $\quad$ $A_{23} = -13$

$a_{31} = 1$, $M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix}$, $A_{31} = (-1)^{3+1} M_{31}$

$A_{31} = -12$

$M_{31} = -12$

$a_{32} = 5$, $M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$, $A_{32} = (-1)^{3+2} M_{32}$

$= (-1)(-22)$

$= 8 - 30$ $\quad$ $A_{32} = 22$

$M_{32} = -22$

$a_{33} = -7$, $M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix}$, $A_{33} = (-1)^{3+3} M_{33}$

$M_{33} = 18$ $\quad$ $A_{33} = 18$

Consider, $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$

$= 2(-12) + (-3)(22) + (5)(18)$

$= -24 - 66 + 90$

$= -90 + 90$

$= 0$

Q. Find the co-factors of the elements of the matrix

$\begin{bmatrix} 2 & 1 & -3 \\ 3 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

Q. Find the adjoint of $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

Q. Find the adjoint of $A = \begin{bmatrix} -5 & 7 \\ -2 & 3 \end{bmatrix}$ and hence ST $A(\text{Adj}A) = |A|I$

Q. Find the adjoint of $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -1 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ and verify

$A(\text{adj} A) = |A| I$.

A_{11} = Cofactor of $2 = +(-13) = -13$

A_{12} = Cofactor of $-1 = -(7) = -7$

A_{13} = Cofactor of $3 = +(9) = 9$

A_{21} = Cofactor of $4 = -(-9) = +9$

A_{22} = cofactor of $-1 = +(3) = 3$

A_{23} = cofactor of $5 = -(5) = -5$

A_{31} = Cofactor of $1 = +(-2) = -2$

A_{32} = Cofactor of $2 = -(-2) = 2$

A_{33} = Cofactor of $3 = +(2) = 2$

$$\text{adj} A = \begin{bmatrix} -13 & -7 & 9 \\ 9 & 3 & -5 \\ -2 & 2 & 2 \end{bmatrix}^T = \begin{bmatrix} -13 & 9 & -2 \\ -7 & 3 & 2 \\ 9 & -5 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 4 & -1 & 5 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 2(-3-10) - (-1)(12-5) + 3(8+1)$$

$$= 2(-13) + 7 + 3(9)$$

$$= -26 + 7 + 27$$

$$|A| = 8$$

$$|A| I = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$A(\text{adj} A) = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -1 & 5 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -13 & 9 & -2 \\ -7 & 3 & 2 \\ 9 & -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -26+7+27 & 18-3-15 & -4-2+6 \\ -52+7+45 & 36-3-25 & -8-2+10 \\ -13-14+27 & 9+6-15 & -2+4+6 \end{bmatrix}$$

$$A(\text{adj}A) = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$\therefore A(\text{adj}A) = |A| I$ (verified)

Singular Matrix :-

A Square matrix A is said to be singular if and only if $|A| = 0$

Non-Singular Matrix :-

A square matrix A is said to be non-singular if and only if $|A| \neq 0$.

Inverse of a matrix :-

Let A be a square matrix of order n and if there exists a matrix B of order $n \times n$ such that $AB = BA = I$, where I is an identity matrix of order n. Then, the matrix B is called the inverse of A and is denoted by $A^{-1} = B$.

For a non-zero square matrix A, A^{-1} exists, whenever A is non-singular $(|A| \neq 0)$ then; $A^{-1} = \dfrac{\text{adj}A}{|A|}$.

NOTE :-

If A and B are non-singular matrices. Then,

(i) $(A^{-1})^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

(iii) $(KA)^{-1} = K^{-1}A^{-1} = (1/K)A^{-1}$ (non-zero scalar K)

(iv) $(A^T)^{-1} = (A^{-1})^T$

Q. Find the inverse of the matrix $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

interchange → only signs change to find adj A

Sol:- $\text{Adj} A = \begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix}$

$A^{-1} = \dfrac{\text{adj}A}{|A|} = \dfrac{1}{-1}\begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix}$

$|A| = -4 + 3$

$= -1$

$= \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

Q. Find the inverse of $\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$

$$A^{-1} = \frac{-1}{2}\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

Q. Find the inverse of $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

$A_{11} = +(-28+30) = +2$ $\quad$ $A_{21} = -(0-6) = 6$

$A_{12} = -(-21-0) = 21$ $\quad$ $A_{22} = +(-7+0) = -7$

$A_{13} = +(-18+0) = -18$ $\quad$ $A_{23} = -(-6+0) = 6$

$A_{31} = +(0+4) = 4$

$A_{32} = -(5+3) = -8$

$A_{33} = +(4+0) = 4$

$$\text{adj} A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix}$$

$$= 1(-28+30) - 0 - 1(-18+0)$$

$$|A| = 2+18 = 20.$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj} A$$

$$= \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Rank of a Matrix

A matrix A is said to be of rank r when

(i) It has atleast one non-zero minor of order r.

(ii) Every minor of order higher than r vanishes.

The rank of a matrix is denoted by $\rho(A)$.

NOTE :-

1. Let A be any square matrix of order 2 and if A is singular matrix i.e., if $|A| = 0$, then rank of the matrix A is less than or equal to 2 $\Rightarrow \rho(A) \leq 2$

2. If A is non-singular matrix of order 2, then rank of matrix A is equal to 2. $\Rightarrow \rho(A) = 2$.

Q. Find the rank of the matrix $\begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$

Consider, $|A| = \begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} = 9 - 15 = -6 \neq 0$

Since $|A| \neq 0$, the rank of the matrix is 2 i.e., $\rho(A) = 2$

Q. Find the rank of the matrix $\begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$

Consider $|A| = \begin{vmatrix} -5 & -7 \\ 5 & 7 \end{vmatrix} = -35 + 35 = 0$

$|A| = 0$ i.e. Singular

$\therefore \rho(A) \neq 2$, but less than 2

Now, consider first order minor

$|-5| = -5 \neq 0$

$\therefore \rho(A) = 1$

Q. Find the rank of the matrix $\begin{bmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{bmatrix}$

A:- $\rho(A) = 3$

Q. Find the rank of the matrix $\begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$

Sol:- let $A = \begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$

Consider, Third order minor

$|A| = \begin{vmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{vmatrix} = 5(16-16) - 3(8-8) + 0$
$= 0$

Since $|A| = 0$, $\rho(A) \neq 3$, but less than 3

Consider, second order minor

$$\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7 \neq 0$$

Since second order minor is not zero.

$\therefore \rho(A) = 2$.

Q. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$

Consider, Third order minors

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 1(12-6) - 2(6-3) - 1(12-12)$$
$$= 6 - 6 - 0$$
$$= 0$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{vmatrix} = 2(-7+6) + 1(-28+12) + 3(12-6)$$
$$= -2 - 16 + 18$$
$$= 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -7 \end{vmatrix} = 1(-28+12) - 2(-14+6) + 3(12-12)$$
$$= -16 + 16 + 0$$
$$= 0$$

$$\begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & -7 \end{vmatrix} = 0$$

Since all third order minors vanishes, $\rho(A) \neq 3$

Consider second order minor $\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 2 + 4 = 6 \neq 0$

$\therefore \rho(A) = 2$.

Consistent system: A system of equations is said to be consistent if one or more solution exists.

Inconsistent system: A system of equations is said to be inconsistent if its soln does not exist.

System of Linear Equations.

A system of equation in which all the unknown quantities appear in the first degree alone is called a linear system of equations.

Consider,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \text{—(1)}$$

Here $a_{11}, a_{12}, a_{13}, \dots a_{mn}$ are known real constants called the co-efficients of $x_1, x_2, \dots, x_n$ and $b_1, b_2, b_3, \dots, b_m$ are constants.

If $b_1 = b_2 = \dots = b_m = 0$, then eqn(1) is called homogeneous system of linear equations.

ex:- $x+2y+3z=0$
$8x+2y+9z=0$
$4x+3y+7z=0$ is a homogeneous system with three unknown variables.

ex:- $3x+4y=8$
$9x+3y=5$ is a non-homogeneous system with two unknown variables.

The system of equation (1) can be written in the matrix form as : $AX = B$

where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

To solve system of linear equations we have two methods

1. Gamer's Rule
2. Matrix method.

Cramer's Rule.

It is one of the methods used to solve a system of equations.

To find the solution of the system $AX = B$

- Find the determinants.
  $D = |A|, D_{x_1}, D_{x_2}, \ldots, D_{x_n}$.
- $X_1 = \frac{D_{x_1}}{D}$; $X_2 = \frac{D_{x_2}}{D}$; ...; $X_n = \frac{D_{x_n}}{D}$, where $|A| \neq 0$

[• no. of eqn's should be same as variables]

1. Illustrate the Gamer's rule with the following system:

$$5x + 3y = 1$$
$$3x + 5y = -9$$

Sol:- $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ -9 \end{bmatrix}$

$|A| = 25 - 9$

$|A| = 16$

Δ or $D = 16$

Δ_x or $D_x = \begin{bmatrix} 1 & 3 \\ -9 & 5 \end{bmatrix}$

$|\Delta_x| / |D_x| = 5 + 27$
$= 32$

$\Delta y = \begin{bmatrix} 5 & 1 \\ 3 & -9 \end{bmatrix} = -45 - 3 = -48$

$\therefore x = \frac{|\Delta x|}{|\Delta|} = \frac{32}{16} = 2$

$y = \frac{\Delta y}{\Delta} = \frac{-48}{16} = -3$

2. Solve using Cramer's rule.

$x + 2y + 3z = 1$

$-x + 2z = 2$

$-2y + z = -2$

Sol:- $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

Determinant of A $= \Delta = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix}$

$\Delta = 1(0+4) - 2(-1-0) + 3(2-0)$

$\Delta = 4 + 2 + 6 = 12$

$\Delta = 12$

$\Delta_x = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ -2 & -2 & 1 \end{vmatrix} = 1(0+4) - 2(2+4) + 3(-4-0)$

$\Delta x = 4 - 12 - 12$

$\Delta x = -20$

$\Delta y = \begin{vmatrix} 1 & 1 & 3 \\ -1 & 2 & 2 \\ 0 & -2 & 1 \end{vmatrix} = 1(2+4) - 1(-1-0) + 3(2-0)$

$\Delta y = 6 + 1 + 6$

$\Delta y = 13$

$\Delta z = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & -2 & -2 \end{vmatrix} = 1(0+4) - 2(2-0) + 1(2-0)$

$\Delta z = 2$

$x = \frac{\Delta x}{\Delta} = \frac{-20}{12}$ $\quad y = \frac{\Delta y}{\Delta} = \frac{13}{12}$ $\quad z = \frac{\Delta z}{z} = \frac{2}{12}$

$x = \frac{-5}{4}, \; y = \frac{13}{12}, \; z = \frac{1}{6}.$

3. Solve using Cramer's Rule

$$4(y-x) = 5z - 22$$
$$3z + 4x = 6y + 2$$
$$z - 3y = 14 - 10x$$

Sol:- Arrange the given system of eqns in standard form,

$$4x - 4y + 5z = 22$$
$$4x - 6y + 3z = 2$$
$$10x - 3y + z = 14$$

$$A = \begin{bmatrix} 4 & -4 & 5 \\ 4 & -6 & 3 \\ 10 & -3 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 22 \\ 2 \\ 14 \end{bmatrix}$$

$x = 2, \; y = 4, \; z = 6.$

4. Solve using cramer's rule.

$$3x + y + z = 3$$
$$2x + 2y + 5z = -1$$
$$x - 3y - 4z = 2.$$

Sol:- $x = 1, y = 1, z = -1$

5. Solving using Cramer's rule.

$$3x + 4y = 7$$
$$7x - y = 6$$

Sol:- $x = 1, y = 1.$

Matrix method

Consider the following system of linear equations

$$a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1$$

$$a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2$$

$$a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3$$

The matrix form of above system of equations is $AX=B$.

where $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$AX=B \Rightarrow X=A^{-1}B, \text{ where } |A|\neq 0.$$

Working Rule :-

Step 1 :- First we have to construct matrix form of the given system of equations.

Step 2 :- Verify $|A|\neq 0$

Step 3 :- Solve $X=A^{-1}B$. Thus, it gives the solution of the given system of equations.

Problems :-

Examine the consistency of the following system of equations.

1. $x+2y=2$

$2x+3y=3$

Above eqn's can be written as $AX=B$

where, $A=\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $B=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $X=\begin{bmatrix} x \\ y \end{bmatrix}$

$|A|=3-4=-1\neq 0$

Since, $|A|\neq 0$, A is non-singular $\therefore A^{-1}$ exists.

Hence, the given system of equations is consistent.

2. Solve by matrix method. $2x - 3y = 1$
$3x - y = 3$

Sol. The given system of equations can be written as $AX = B$

where $A = \begin{bmatrix} 2 & -3 \\ 3 & -1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 2 & -3 \\ 3 & -1 \end{vmatrix} = -2 + 9 = 7 \neq 0$

A is non-singular, A^{-1} exists.

$\therefore$ Given system is consistent and has a unique solⁿ.

$\Rightarrow X = A^{-1}B$.

$\text{adj}A = \begin{bmatrix} -1 & 3 \\ -3 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{adj}(A)$

$A^{-1} = \frac{1}{7} \begin{bmatrix} -1 & 3 \\ -3 & 2 \end{bmatrix}$

$X = A^{-1}B = \frac{1}{7} \begin{bmatrix} -1 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$X = \frac{1}{7} \begin{bmatrix} -1+9 \\ -3+6 \end{bmatrix}$

$X = \frac{1}{7} \begin{bmatrix} 8 \\ 3 \end{bmatrix}$.

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8/7 \\ 3/7 \end{bmatrix}$ $\therefore x = 8/7$ and $y = 3/7$.

Q3. Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Sol.- The given equation can be written as $AX = B$

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$|A| = 3(2-3) + 2(4+4) + 3(-6-4)$$
$$= 3(-1) + 2(8) + 3(-10)$$
$$= -3 + 16 - 30$$
$$|A| = -17 \neq 0$$

A is non-singular & A^{-1} exists.

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$adj A = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$X = A^{-1} B$$

$$X = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-17} \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \therefore x = 1, y = 2, z = 3.$$

Q4. Solve by matrix method

a) $5x+2y=4$
$7x+3y=5$

Sol: $x=2, y=-3$

b) $2x_1+3x_2=3$
$5x_1+4x_2=11$

Sol:- $x_1=3, x_2=-1$

Eigen-Values and Eigen Vectors.

The Eigenvalues of matrix are scalars by which some vectors change when the matrix (transformation) is applied to it.

If A is a square matrix and X is a column vector such that:

$$AX = \lambda X$$

X - Eigen-vector of A

λ - Eigenvalue of A.

From the definition of eigen values, if λ is an eigen value of a Square matrix A, then

$$AX = \lambda X$$

If I is the identity matrix of the same order as A, Then we can write

$$AX = \lambda (IX) \qquad (\text{Because } X = IX)$$

$$AX - \lambda (IX) = 0$$

$$X(A - \lambda I) = 0$$

It has non-trivial sol^n only when the determinant of the coefficient matrix is 0.

i.e., $|A - \lambda I| = 0$

This eq^n is called the characteristic equation.

Definition :-

Let $A = [a_{ij}]_{m \times n}$ be a square matrix and λ be a scalar. Then, the matrix $[A - \lambda I]$ is called the characteristic equation of A and the roots of this equation are called the characteristic root or characteristic value or Eigen values.

Suppose there exists a non-zero column matrix X such that $AX = \lambda X$. Then X is called an Eigenvector of A and λ is called the corresponding Eigen value of A.

If A is a square matrix of order 'n' then the characteristic matrix equation is given by,

$$[A - \lambda I][X] = [0]$$

where λ is a scalar
I is a identity matrix of order 'n'.
X is a column matrix of variables of order $n \times 1$.

Properties of Eigen values and Eigen vectors.

1. Sum of the Eigenvalues of a squarematrix is equal to the sum of principal diagonal elements.
2. The product of eigenvalues of a matrix A is equal to its determinant value.
3. A square matrix A and its transpose A^T have the same eigen values.
4. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of matrix A then, (a) The A^{-1} has the eigen values $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$

   (b) The matrix A^2 has the eigenvalues $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$

(c) The matrix KA has the eigen values $k\lambda_1, k\lambda_2, \ldots k\lambda_n$, where k is a non-zero scalar.

5. Eigen vector X of a matrix is not unique.

<u>Problems</u>

Find the Eigenvalues and Eigenvectors of the matrices

(i) $A = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}$

Sol:- Given $A = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}$

The characteristic eq^n of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & 8 \\ -2 & 7-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)(7-\lambda) + 16 = 0$$

$$-21 + 3\lambda - 7\lambda + \lambda^2 + 16 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 5, -1$$

-5

-5 1

-4

$\therefore$ The eigen values are 5 and -1.

From characteristic matrix equation,

$$[A - \lambda I][X] = 0$$

$$\begin{bmatrix} -3-\lambda & 8 \\ -2 & 7-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (-3-\lambda)x + 8y &= 0 \\ -2x + (7-\lambda)y &= 0 \end{aligned}\right\} \rightarrow (1)$$

Case(i):- For $\lambda = 5$

eqn(1) becomes,

$$(-3-5)x + 8y = 0$$
$$-2x + (7-5)y = 0$$

$$\rightarrow \quad -8x + 8y = 0$$
$$-2x + 2y = 0$$

$$\rightarrow \quad 8x = 8y$$
$$\frac{x}{8} = \frac{y}{8}$$
$$\frac{x}{1} = \frac{y}{1}$$

$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the Eigenvector corresponding to $\lambda = 5$.

Case(ii):- For $\lambda = -1$

eqn(1) becomes,

$$(-3-(-1))x + 8y = 0$$
$$-2x + (7-(-1))y = 0$$

$$\Rightarrow \quad -2x + 8y = 0$$
$$-2x + 8y = 0$$

$$\Rightarrow \quad 2x = 8y$$
$$\frac{x}{8} = \frac{y}{2} \Rightarrow \frac{x}{4} = \frac{y}{1}$$

$X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -1$.

(ii) $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $A - \lambda I = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)(4-\lambda) - (-2) = 0$

$4 - \lambda - 4\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 3)(\lambda - 2) = 0$

$\lambda = 2, 3$

6, -3, -2, -5

$\therefore$ Eigenvalues are 2 and 3

Characteristic matrix equation, $[A - \lambda I][X] = [0]$

$\begin{bmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{aligned} (1-\lambda)x - 2y &= 0 \\ x + (4-\lambda)y &= 0 \end{aligned}\right\} \rightarrow (1)$

Case (i) :- For $\lambda = 2$

eq^n (1) becomes, $\left[\begin{aligned} -x - 2y &= 0 \\ x + 2y &= 0 \end{aligned}\right.$

$x = -2y$

$\frac{x}{-2} = \frac{y}{1}$

$X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is the Eigenvector corresponding to $\lambda = 2$.

Case (ii):- for $\lambda = 3$

eq^n ① becomes,

$$-2x - 2y = 0$$
$$x + y = 0$$

$$-2x = 2y$$
$$-\frac{x}{2} = \frac{y}{2}$$
$$-\frac{x}{1} = \frac{y}{1}$$

$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the Eigenvector corresponding to $\lambda = 3$.

(iii) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol:- Characteristic eq^n of A is, $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 1, 6$$

6

-6 $\quad$ -1

-7

$\therefore$ Eigenvalues are 1 and 6.

Characteristic matrix equation,

$$[A-\lambda I][X]=0$$

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned}(5-\lambda)x+4y&=0\\ x+(2-\lambda)y&=0\end{aligned}\right\}\rightarrow (1)$$

Case (i):-

for $\lambda=1$

eq^n (1) becomes,

$$(5-1)x+4y=0$$
$$x+(2-1)y=0$$

$$\Rightarrow 4x+4y=0$$
$$x+y=0$$

$$\frac{x}{1}=\frac{-y}{1}.$$

$X_1=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors corresponding to $\lambda=1$

Case (ii) :-

for $\lambda=6$,

eq^n becomes, $-x+4y=0$
$x-4y=0$

$$x=4y$$
$$\frac{x}{4}=\frac{y}{1}$$

$X_2=\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ are Eigenvectors corresponding to $\lambda=6$.

(iv) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

Sol:- Characteristic eqn of A, $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 12 = 0$$

$$2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda - 5)(\lambda + 2) = 0$$

$$\lambda = 5, -2$$

-10

$-5 \quad 2$

-3

$\therefore$ Eigen values are 5 and -2.

The characteristic matrix eqn,

$$[A - \lambda I][X] = 0$$

$$\begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (1-\lambda)x + 4y &= 0 \\ 3x + (2-\lambda)y &= 0 \end{aligned}\right\} \rightarrow ①$$

Case (i) :- For $\lambda = -2$

$$+3x + 4y = 0$$
$$3x + 4y = 0$$

$$\Rightarrow 3x = -4y$$

$$\frac{x}{-4} = \frac{y}{3}$$

$X_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is the corresponding eigenvector to $\lambda = -2$.

Case (ii):- For $\lambda = 5$

$$-4x + 4y = 0$$
$$3x - 3y = 0$$

$$\Rightarrow 4x = 4y$$

$$\frac{x}{4} = \frac{y}{4}$$

$$\frac{x}{1} = \frac{y}{1}$$

$\Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 5$.

Extra Questions

1. Solve the following by Cramer's rule.

(a) $3x + 5y = 9$
$2y - 5x = 16$

(b) $2x + 3y - 1 = 0$
$3x - y + 2 = 0$.

2. Find the rank of the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$

3. Examine the consistency of the system of equations.

(i) $3x - y - 2z = 2$
$2y - z = -1$
$3x - 5y = 3$.

4. Solve by matrix method:

$x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$.

5. Find the inverse of $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 1 & 1 \\ 1 & -2 & 3 \end{bmatrix}$