UNIT-3,G- GAME THEORY

Game Theory was developed for decision making under conflicting situations where there are one or more opponents (players). Game theory provides optimal solutions to games, assuming that each of the players wants to maximize his profit or minimize his loss.

Basic Terminologies.

1. **Player**: Each participant of a game is called a player.

2. **Strategy**: The strategy of a player is the predetermined rule by which a player decides his course of action from the list of courses of action during the game.
   A strategy may be of two types :-

   - **Pure strategy** :- It is a decision, in advance of all plays, always to choose a particular course of action.

   - **Mixed strategy** :- It is a decision, in advance of all plays, to choose of course of action for each play in accordance with some particular probability distribution.

3. **Optimal strategy** :- The course of action which maximizes the profit of a player or minimizes his loss is called an optimal strategy.

4. **Pay off** :- The outcome of playing a game is called pay off.

5. Pay off matrix :- When the players select their particular strategies, the payoffs can be represented in the form of a matrix called the payoff matrix.

6. Saddle point :- A saddle point is an element of the pay off matrix, which is both the smallest element in its row and the largest element in its column. Furthermore, the saddle point is also regarded as an equilibrium point in the theory of games.

7. Value of the game :- It refers to the expected outcome per play when players follow their optimal Strategy.

TWO PERSON ZERO-SUM GAME

A game with only two players is called a two-person zero-sum game if the losses of one player are equivalent to the gains of the other so that the sum of their net gains is zero. This game also known as rectangular game.

① Find the Optimum strategies of the players in the following games.

Player B

Player A		B_1	B_2	B_3
	A_1	1	3	1
	A_2	0	-4	-3
	A_3	1	5	-1

Sol:

		B_1	B_2	B_3	Row minima
Player A	A_1	1	3	1	$\boxed{1}$
	A_2	0	-4	-3	-4
	A_3	1	5	-1	-1
Column maxima		$\boxed{1}$	5	$\boxed{1}$	

Maximin = Minimax

Minimax = 1
Maximin = 1 $\Rightarrow$ Saddle point
So = A_1B_1
or A_1B_3

Q2. Find the Optimum strategies of the players in the following games.

		Player B		
		1	2	3
Player A	1	25	20	35
	2	50	45	55
	3	58	40	42

Sol:-

		Player B 1	2	3	Row minima	Maximin
Player A	1	25	20	35	20	= 45
	2	50	45	55	$\boxed{45}$	
	3	58	40	42	40	
Column maxima		58	$\boxed{45}$	55		

Minimax = 45

Maximin = Minimax

45 = 45

Value of the game, $V = 45$

$\therefore$ The game has a saddle point at the cell corresponding to (R_2, C_2)

Optimal Probabilities

$A[P_1, P_2, P_3] = A[0, 1, 0]$

$B[q_1, q_2, q_3] = B[0, 1, 0]$

Q3. Determine the optimal strategies for each player in the following game. (with saddle point).

Player T

$$\begin{array}{cc} & \begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \end{array} \\ \text{Player S} \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} & \begin{bmatrix} -5 & 2 & 0 & 7 \\ 5 & 6 & 4 & 8 \\ 4 & 0 & 2 & -3 \end{bmatrix} \end{array}$$

Sol:- We apply the maximin - minimax principle to analyze the game.

Player T

		T_1	T_2	T_3	T_4	Row minimum
Player S	S_1	-5	2	0	7	-5
	S_2	5	6	[(4)]	8	[4]
	S_3	4	0	2	-3	-3
Column maximum		5	6	(4)	8	

Select minimum from the maximum of columns

Minimax = (4)

Select maximum from the minimum of rows,

Maximin = [4]

Here, Minimax = Maximin = 4.

∴ This game has a saddle point & value of the game is 4.

The Optimal Strategies for both players are,

Player A will always adopt strategy 2

Player B will always adopt strategy 3

Q.4. For the game with pay off matrix,

$$\begin{array}{cc} & \text{Player B} \\ \text{player A} & \begin{vmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{vmatrix} \end{array}$$

Determine the best strategies for players A & B and also the value of the game. Is this game
(i) fair (ii) Strictly determinable?

Sol.:

$$\begin{array}{ccc} & \text{Player B} & \text{Row minimum} \\ \text{Player A} & \begin{vmatrix} -1 & 2 & [(-2)] \\ 6 & 4 & -6 \end{vmatrix} & \begin{matrix} [-2] \\ -6 \end{matrix} \\ \text{Column maximum} & \begin{matrix} 6 & 4 & (-2) \end{matrix} & \end{array}$$

Maximin = −2

Minimax = −2

Maximin = Minimax = −2

∴ We have have saddle point $S_0 = A_1B_3$

Value of the game = −2

The game is not fair but strictly determinable

Games without saddle points, Mixed strategies.

1. Solution of 2x2 games without Saddle point.

For any 2x2 two-person zero-sum game without any saddle point having the pay off matrix for player A is.

$$\begin{array}{cc} & \begin{matrix} B_1 & B_2 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \end{matrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{array}$$

The optimum mixed strategies

$$S_A = \begin{vmatrix} A_1 & A_2 \\ p_1 & p_2 \end{vmatrix} \text{ and } S_B = \begin{vmatrix} B_1 & B_2 \\ q_1 & q_2 \end{vmatrix}$$

where $p_1 = \dfrac{a_{22} - a_{21}}{\lambda}$ and $p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$

$q_1 = \dfrac{a_{22} - a_{12}}{\lambda}$ and $q_1 + q_2 = 1 \Rightarrow q_2 = 1 - p_2$

where $\lambda = (a_{11} + a_{22}) - (a_{12} + a_{21})$

and $v = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$.

Q. Solve the following game without saddle point.

$$\begin{array}{cc} & B \\ A & \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} \end{array}$$

Sol:- Let the given pay-off matrix be,

$$\begin{array}{ccc} & B_1 & B_2 \\ A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

The optimum mixed strategies,

Strategy for A, $S_A = \begin{vmatrix} A_1 & A_2 \\ p_1 & p_2 \end{vmatrix}$

Strategy for B, $S_B = \begin{vmatrix} B_1 & B_2 \\ q_1 & q_2 \end{vmatrix}$

Now, $\lambda = (a_{11} + a_{22}) - (a_{12} + a_{21})$

$\lambda = (2+1) - (5+4) = 3 - 9 = -6 \Rightarrow \boxed{\lambda = -6}$

$$P_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{1-4}{-6} = \frac{-3}{-6} = \frac{1}{2}$$

$$P_2 = 1 - P_1 = 1 - 1/2 = 1/2$$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{1-5}{-6} = \frac{-4}{-6} = \frac{2}{3}$$

$$q_2 = 1 - q_1 = 1 - \frac{2}{3} = \frac{1}{3}$$

Value of the game, $\frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda} = \frac{2-20}{-6} = 3$

$$S_A = \begin{vmatrix} A_1 & A_2 \\ 1/2 & 1/2 \end{vmatrix} \qquad S_B = \begin{vmatrix} B_1 & B_2 \\ 2/3 & 1/3 \end{vmatrix}$$

Q2. In a game of matching coins with two players, suppose A wins one unit value when there are two heads, wins nothing when there are 2 tails and looses 1/2 unit value when there are one head & one tail. Determine the payoff matrix, the best strategy for each players, and the value of the game.

Sol:

Formulation

$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} H & T \end{array} \\ A & \begin{array}{c} H \\ T \end{array} & \begin{vmatrix} 1 & -1/2 \\ -1/2 & 0 \end{vmatrix} \end{array}$$

Let the pay-off matrix be,

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{array}$$

The Optimum mixed strategies,

$S_A = \begin{vmatrix} A_1 & A_2 \\ p_1 & p_2 \end{vmatrix}$ and $S_B = \begin{vmatrix} B_1 & B_2 \\ q_1 & q_2 \end{vmatrix}$

Now, $\lambda = (a_{11} + a_{22}) - (a_{12} + a_{21})$

$\lambda = (1+0) - (-1/2 - 1/2)$

$\lambda = 1+1$

$\lambda = 2.$

$p_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{0-(-1/2)}{2} = \frac{1}{4}$

$p_2 = 1 - p_1 = 1 - \frac{1}{4} = \frac{3}{4}$

$q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{0-(-1/2)}{2} = \frac{1}{4}$

$q_2 = 1 - q_1 = 1 - 1/4 = 3/4$

Value of the game, $v = \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$

$= \frac{1(0) - (-1/2)(-1/2)}{2}$

$v = \frac{-1/4}{2} = -\frac{1}{8}$

Strategy for $A = (1/4, 3/4)$

Strategy for $B = (1/4, 3/4)$

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Q3. Solve the following 2×2 game

$$\begin{array}{cc} & B \\ A & \begin{vmatrix} 2 & 5 \\ 7 & 3 \end{vmatrix} \end{array}$$

Sol:-

$$\begin{array}{ccc} & B & \text{Row minima} \\ A & \begin{vmatrix} 2 & 5 \\ 7 & 3 \end{vmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \\ \text{Column maxima} & 7 \quad 5 & \end{array}$$

Minimax = 5

Maximin = 3

$\overline{V} \neq \underline{V} \Rightarrow$ No saddle point.

Let B choose 2, 5 with probability $x, 1-x$
& 7, 3 with probability $x, 1-x$

$$2x+5(1-x) = 7x+3(1-x)$$
$$2x+5-5x = 7x+3-3x$$
$$5-3x = 5x+3$$
$$5-3 = 5x+3x$$
$$8x = 2$$
$$x = 2/8$$
$$x = 1/4$$
$$\therefore 1-x = 1-1/4 = 3/4.$$

Strategy for $B = \left(\frac{2}{7}, \frac{5}{7}\right)$

Let A choose 2, 7 with probability $y, 1-y$
& 5, 3 with probability $y, 1-y$

Then $2y+7(1-y) = 6y+3(1-y)$

$$2y+7-7y = 5y+3-3y$$
$$7-5y = 2y+3$$
$$7-3 = 2y+5y$$
$$7y = 4$$
$$y = \frac{4}{7}$$
$$\therefore 1-y = 1-\frac{4}{7} = \frac{3}{7}$$

$\therefore$ Stratigy for A $= \left(\frac{4}{7}, \frac{3}{7}\right)$

Value of the game $= 2\times\frac{4}{7} + 7\times\frac{3}{7}$

$$= \frac{8}{7} + \frac{21}{7} = \frac{29}{7}.$$

Q4. Solve the following 2×2 game,

$$\begin{array}{cc} & B \\ A & \begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix} \end{array}$$

$\therefore$ Let B choose 5, 1 with probability $x, 1-x$
& 3, 4 with poobability $x, 1-x$.

$$5x+(1-x) = 3x+4(1-x)$$
$$5x+1-x = 3x+4-4x$$
$$4x+1 = 4-x$$
$$5x = 3$$
$$x = 3/5$$
$$1-x = 1-3/5 = 2/5$$

Stratigy for B $= (3/5, 2/5)$

let A choose 5, 3 with probability $y, 1-y$
& 1, 4 with probability $y, 1-y$.

$$5y + 3(1-y) = y + 4(1-y)$$

$$5y + 3 - 3y = y + 4 - 4y$$

$$2y + 3 = 4 - 3y$$

$$5y = 1$$

$$y = 1/5$$

$$1 - y = 1 - 1/5 = 4/5$$

Strategy for A $= (1/5, 4/5)$

Value of the game $= \cancel{5} \times \frac{3}{\cancel{5}} + 1 \times \frac{2}{5}$

$$= 3 + \frac{2}{5} = 17/5.$$

DOMINANCE PROPERTY FOR $m \times n$ GAME.

In some games, it is possible to reduce the size of the payoff matrix by eliminating rows (or columns) which are dominated by other rows (or columns) respectively.

Dominance property for rows:-

If all the entries in a row ~~should be~~ is lesser than or equal to the corresponding entries of another row, then that row can be deleted.

$X \leq Y$.

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Dominance for columns.

If all the entries in a column should be is greater than or equal to the corresponding entries of another column, then that column can be deleted.

$X \geq Y$.

Q. Solve the following game using dominance property

$$\begin{array}{cc|ccc} & & & B & \\ & & I & II & III \\ A & I & 1 & 7 & 2 \\ & II & 6 & 2 & 7 \\ & III & 6 & 1 & 6 \end{array}$$

A: Using dominance property.

row-3 is dominated by row-2, so row-3 is deleted.
$(6 \leq 6, 1 \leq 2, 6 \leq 7)$

$$\begin{array}{cc|ccc} & & & B & \\ & & I & II & III \\ & I & 1 & 7 & 2 \\ A & II & 6 & 2 & 7 \end{array}$$

Column-3 is dominated by column-1, so column-3 is deleted. $(2 \geq 1, 7 \geq 6)$

$$\begin{array}{cc|cc|c} & & \multicolumn{2}{c}{B} & \\ & & I & II & \text{row minimum} \\ A & I & 1 & 7 & 1 \\ & II & 6 & 2 & 2 \\ \text{Column maximum} & & 6 & 7 & \end{array}$$

Maximin = 2

Minimax = 6

Maximin $\neq$ minimax $\Rightarrow$ No Saddle point

If B chooses 1, 7 with probabilities $x, (1-x)$
& 6, 2 with probabilities $x, (1-x)$

We have,
$$x + 7(1-x) = 6x + 2(1-x)$$
$$x + 7 - 7x = 6x + 2 - 2x$$
$$7 - 6x = 4x + 2$$
$$10x = 5$$
$$\boxed{x = 1/2}$$
$$1 - x = 1 - 1/2 = 1/2$$

$\therefore$ Strategy for B, $S_B = (1/2, 1/2, 0)$

If A chooses 1, 6 with probabilities $y, (1-y)$
& 7, 2 with probabilities $y, (1-y)$

We have
$$y + 6(1-y) = 7y + 2(1-y)$$
$$y + 6 - 6y = 7y + 2 - 2y$$
$$6 - 5y = 2 + 5y$$
$$10y = 4$$
$$y = \frac{4}{10}$$
$$\boxed{y = 2/5}$$
$$1 - y = 1 - 2/5 = 3/5$$

$\therefore$ Strategy for A, $S_A = (2/5, 3/5, 0)$

Value of the game $= 1 \cdot 1/2 + 7 \cdot 1/2 = 4$.

Q2. Using Dominance property solve

$$\begin{array}{c c c} & & \text{Player B} \\ & & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} -5 & 3 & 2 & 20 \\ 5 & 5 & 4 & 6 \\ -2 & -2 & 0 & -5 \end{bmatrix} \end{array}$$

A. Using dominance property,

Row III is dominated by row II
Col II is dominated by column I.
⇒ Omit row III, column II.

$$\begin{array}{c c} & \begin{array}{ccc} B_1 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -5 & 2 & 20 \\ 5 & 4 & 6 \end{bmatrix} \end{array}$$

Column IV is dominated by column I
⇒ Omit column IV

$$\begin{array}{c c c} & \begin{array}{cc} B_1 & B_3 \end{array} & \text{Row minimum} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -5 & 2 \\ 5 & \boxed{4} \end{bmatrix} & \begin{array}{c} -5 \\ 4 \end{array} \\ \text{Column maximum} & \begin{array}{cc} 5 & \boxed{4} \end{array} & \end{array}$$

Maximin = 4
Minimax = 4

Saddle point = A_2B_3
Value of the game = 4.

Q. A & B play a game in which each has three coins, a 5p, a 10p and a 20p. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, A wins B's coins. If the sum is even B wins A's coins. Find the best strategy for each player & the value of the game.

Sol:- Formulation of the game.

$$\begin{array}{cc|ccc|} & & & B & \\ & & 5p & 10p & 20p \\ \hline & 5p & -5 & 10 & 20 \\ A & 10p & 5 & -10 & -10 \\ & 20p & 5 & -20 & -20 \end{array}$$

Odd + Odd = Even → B wins
Odd + Even = Odd → A wins
Even + Odd = Odd → A wins
Even + Even = Even → B wins

Row-3 is dominated by Row 2, Omit Row-3

$$\begin{array}{c|ccc|} & 5p & 10p & 20p \\ \hline 5p & -5 & 10 & 20 \\ 10p & 5 & -10 & -10 \end{array}$$

Col-3 is dominated by col-2, Omit col-3

$$\begin{array}{c|cc|} & 5p & 10p \\ \hline 5p & -5 & 10 \\ 10p & 5 & -10 \end{array}$$

$\lambda = (a_{11}+a_{22}) - (a_{12}+a_{21}) = -30$

$p_1 = \dfrac{a_{22}-a_{21}}{\lambda} = \dfrac{1}{2}$ $\quad p_2 = 1/2$, $S_A = (1/2, 1/2)$

$q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{-10-10}{-30} = \frac{2}{3}$

$q_1 = \frac{2}{3}$; $q_2 = 1/3$, $S_B = (2/3, 1/3, 0)$

Value of the game $= \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda} = \frac{50-50}{\lambda} = 0.$

GRAPHICAL METHOD FOR 2×n or m×2 games.

Step 1:- Reduce the size of the payoff matrix of player A by applying dominance property, if it exists.

Step 2:- Let x be the probability of selection of Alternative 1 by player A and $1-x$ be the prob. of selection of Alternative 2 by player B. Derive the expected gain function of player A with respect to each of the alternatives of player B.

Step 3:- Find the value of the gain, when $x=0$ and $x=1$

Step 4:- Plot the gain functions on a graph by assuming a suitable scale.

Step 5:- Find the highest intersection point in the lower boundary of the graph → Maximin point.

Step 6:- If the no. of lines passing through the maximin point is only two, form a 2×2 pay-off matrix
- Go to step 8 ; Otherwise go to step 7.

Step 7:- Identify any two lines with opposite slopes passing through that point. Then form a 2×2 payoff matrix

Step 8:- Solve the 2×2 game using regular method & find the strategies for player A & B and also value of the game.

2×m - Lower envelope
m×2 - Upper envelope.

Q1. Solve the following 2×4 game graphically.

$$\begin{array}{c|cccc|} & & \text{Player B} & & \\ \text{Player A} & 1 & 0 & 4 & -1 \\ & -1 & 1 & -2 & 5 \end{array}$$

Sol:

$$\begin{array}{cc|cccc|c} & & (1) & (2) & (3) & (4) & \\ & & & \text{Player B} & & & \\ \text{Player A} & 1 & 1 & 0 & 4 & -1 & x \\ & 2 & -1 & 1 & -2 & 5 & 1-x \end{array}$$

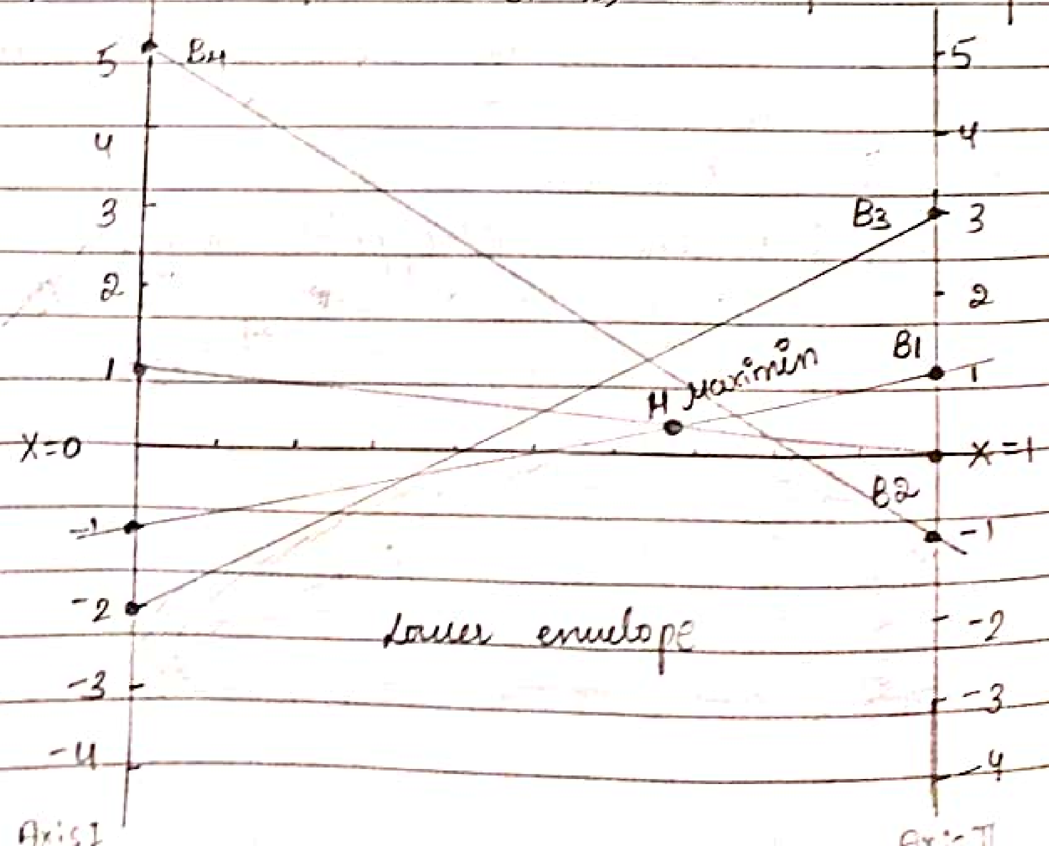
No, course of action dominates the other.

$\therefore$ Take $x, 1-x$ be the probabilities of selected strategy.

Expected payoff function & Gain of player A

B's alternative	A's expected Payoff function	A's expected gain	
		$x=0$	$x=1$
1	$x+(1-x)(-1) = 2x-1$	-1	1
2	$0+(1-x) = 1-x$	1	0
3	$4x-2(1-x) = 5x-2$	-2	3
4	$-x+5(1-x) = 5-6x$	5	-1

(OR) Directly plot



Maximum lines B_1, B_2.

$$\begin{array}{c|cc|c} & B & & \text{Row minimum} \\ A & 1 & 0 & 0 \\ & -1 & 1 & -1 \\ \hline \text{Col max} & 1 & 1 & \end{array}$$

Minimax ≠ Maximin

⇒ No saddle point.

$\lambda = (a_{11}+a_{22}) - (a_{12}+a_{22})$

$= (1+1) - (-1+0)$

$= 2+1 = 3$

$\lambda = 3$

$p_1 = \dfrac{a_{22}-a_{21}}{\lambda} = \dfrac{1-(-1)}{3} = \dfrac{2}{3}$

$q_1 = \dfrac{a_{22}-a_{12}}{\lambda} = \dfrac{1-0}{3} = \dfrac{1}{3}$

$p_2 = 1 - \dfrac{2}{3} = \dfrac{1}{3}$

$q_2 = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$S_A = (2/3, 1/3, 0)$

$S_B = \left(\dfrac{1}{3}, \dfrac{2}{3}, 0\right)$

$$\text{Value of game} = \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$$

$$= \frac{1(1) - 0(-1)}{3}$$

$$V = \frac{1}{3} = 0.33\bar{3}$$

Q. Using graphical method, solve the rectangular game whose payoff matrix for player A is

$$\begin{vmatrix} 2 & -1 & 5 & -2 & 6 \\ -2 & 4 & -3 & 1 & 0 \end{vmatrix}$$

Sol:-

$$\begin{array}{c|ccccc|}
 & B_1 & B_2 & B_3 & B_4 & B_5 \\
\hline
A_1 & 2 & -1 & 5 & -2 & 6 \\
A_2 & -2 & 4 & -3 & 1 & 0 \\
\hline
 & 0 & 3 & 2 & 1 & 6
\end{array}$$

col 5 dominates col 1
col 2 dominates col 4

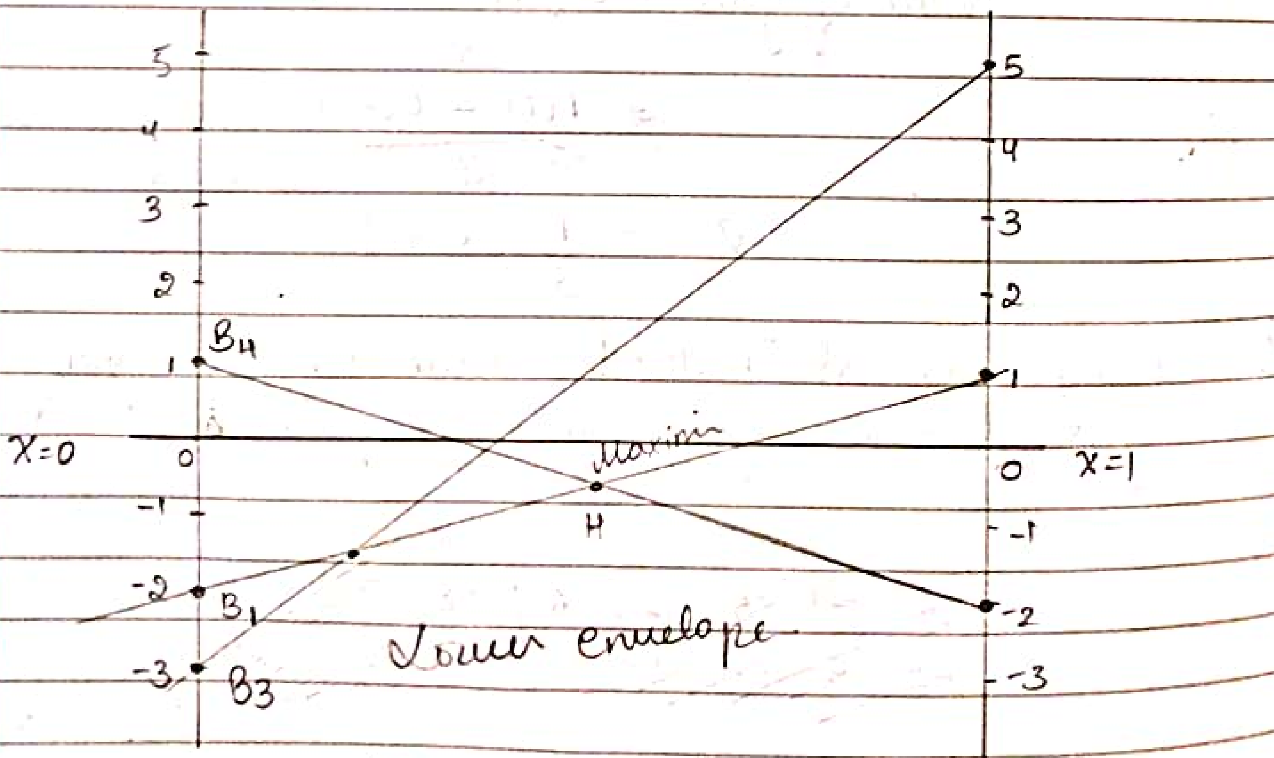
Reduced pay off matrix is,

$$\begin{array}{c|ccc|c}
 & B_1 & B_3 & B_4 & \\
A_1 & 2 & 5 & -2 & x \\
A_2 & -2 & -3 & 1 & 1-x
\end{array}$$

Take x, $1-x$ as probabilities of selected strategies.

Expected pay-off function & Gain of player A

B's Alternative	A's expected pay-off function	A's expected gain $X=0$	$X=1$
1	$2x-2(1-x)=3x-2$	-2	1
3	$5x-3(1-x)=8x-3$	-3	5
4	$-2x+1-x=-3x+1$	1	-2



Maximum lines B_1, B_4

$$
\begin{array}{c|cc|}
 & B_1 & B_4 \\
\hline
A_1 & 2 & -2 \\
A_2 & -2 & 1
\end{array}
$$

$$
\begin{aligned}
\lambda &= (a_{11}+a_{22}) - (a_{12}+a_{21}) \\
&= (2+1) - (-2-2) \\
&= 3 - (-4) \\
&= 7.
\end{aligned}
$$

$$p_1 = \frac{a_{22}-a_{21}}{\lambda} = \frac{1-(-2)}{7} = \frac{3}{7}$$

$$p_2 = 1 - \frac{3}{7} = \frac{4}{7}$$

$$S_A = \left(\frac{3}{7}, 0, 0, \frac{4}{7}, 0\right)$$

$$q_1 = \frac{a_{22}-a_{12}}{\lambda} = \frac{1-(-2)}{7} = \frac{3}{7}$$

$$q_2 = 1 - \frac{3}{7} = \frac{4}{7}$$

$$S_B = \left(\frac{3}{7}, 0, 0, \frac{4}{7}, 0\right)$$

Value of the game $= \dfrac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$

$$= \frac{2(1) - (-2)(-2)}{7}$$

$$V = \frac{2-4}{7} = \frac{-2}{7}$$

Q.3 Using graphical method, solve the rectangular game where payoff matrix for players

		Player B I	II	III	IV
Player A	1	18	4	6	4
	2	6	2	13	7
	3	11	5	17	3
	4	7	6	12	2

Minimax ≠ Maximin & No Saddle point

Sol:

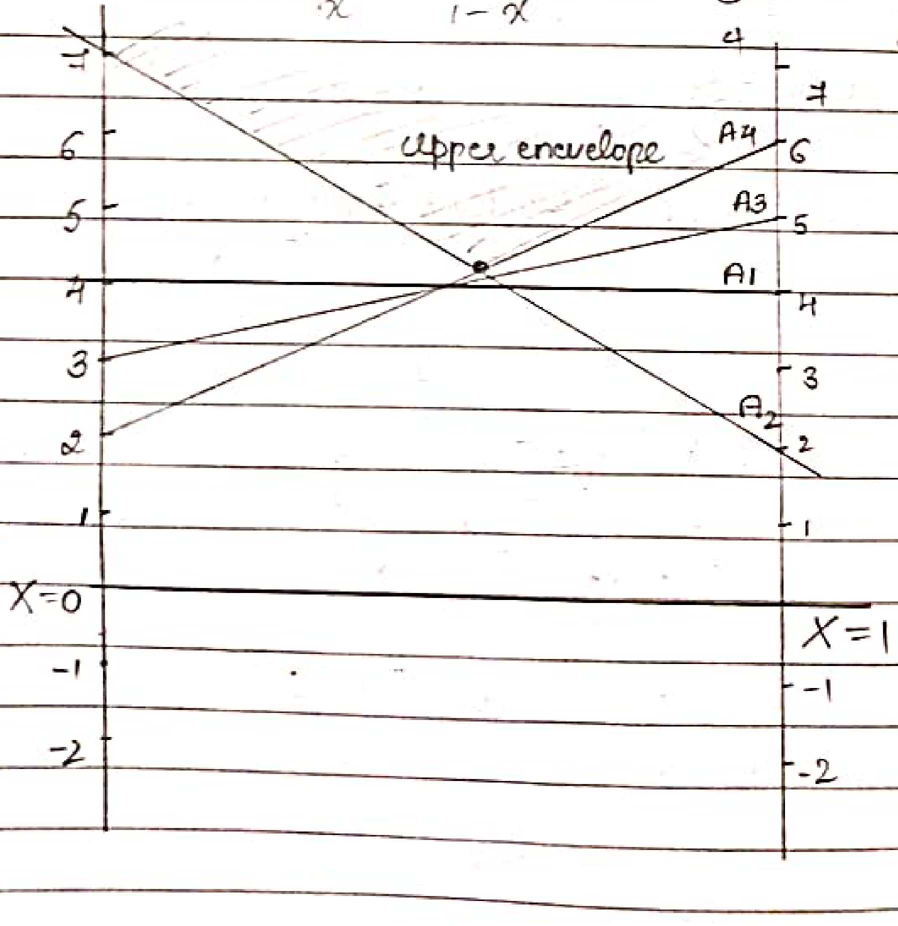
Col I is dominated by col II

col III is dominated by col IV

Omit col I, col III

		Player B II	IV
Player A	A_1	4	4
	A_2	2	7
	A_3	5	3
	A_4	6	2
		x	$1-x$

A's Alternative	B's expected pay off function	$x=1$
1	$4x+4(1-x)$	4
2	$2x+7(1-x)$	2
3	$5x+3(1-x)$	5
4	$6x+2(1-x)$	6



Minimax point is A_2A_4

Q4.

∴ Minimum of the upper envelope is

$$\begin{array}{c|cc|} & \text{II} & \text{IV} \\ \hline 2 & 2 & 7 \\ 4 & 6 & 2 \end{array}$$

Maximin ≠ Minimax
⇒ No Saddle point.

$$\lambda = (a_{11}+a_{22}) - (a_{12}+a_{21})$$
$$= (2+2) - (6+7)$$
$$\lambda = 4 - 13 = -9$$

$$p_1 = \frac{a_{22}-a_{21}}{\lambda} = \frac{2-6}{-9} = \frac{-4}{-9}$$

$p_1 = \frac{4}{9}$; $p_2 = 1 - p_1 \Rightarrow p_2 = \frac{5}{9}$

$S_A = \left(\frac{4}{9}, \frac{5}{9}\right) \Rightarrow S_A = \left(0, \frac{4}{9}, 0, \frac{5}{9}\right)$

$$q_1 = \frac{a_{22}-a_{12}}{\lambda} = \frac{2-7}{-9} = \frac{-5}{-9}$$

$q_1 = \frac{5}{9}$; $q_2 = 1 - q_1 \Rightarrow q_2 = 1 - \frac{5}{9}$

$q_2 = \frac{4}{9}$

$S_B = \left(\frac{5}{9}, \frac{4}{9}\right) \Rightarrow S_B = \left(0, \frac{5}{9}, 0, \frac{4}{9}\right)$

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda} = \frac{2\cdot 2 - 7\cdot 6}{-9} = \frac{4-42}{-9} = \frac{+38}{+9}$$

$V = \frac{38}{9}$

4 Using the Principle of dominance, solve the following game

$$\begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

Sol:- value of game = 2

Unit-3 ; C2 - SEQUENCING PROBLEMS

Sequencing problems are a class of optimization problems in Operations Research that deal with determining the optimal order in which a set of tasks or jobs should be performed.

Basic Terminologies

- Processing time :- This is the time required by each job on each machine.

- Processing order :- This refers to the order (sequence) in which machines are required for completing the job.

- Idle time on a machine :- This is the time during which a machine does not have a job to process.

- Total elapsed time :- This is the time interval between starting the first job & completing the last job, including the idle time (if any), in a particular order by the given set of machines.

- No passing rule :- This means that the passing is not allowed, i.e., the same order of jobs is maintained over each machine. If 'n' jobs are to be processed through two machines A & B in the order AB, then this means that each job will go to machine A first and then to B.

Principal Assumptions

The processing time on each machine is known

No machine can process more than one operation at a time.

Each operation once started must be performed till completion

The time required to complete the job is independent of the order of jobs in which they are to be processed.

Types of Job sequencing problem.

n-jobs are to be processed on 2 machines.

n-jobs are to be processed on 3 machines

n-jobs are to be processed on m machines

2 jobs are to be processed on 'm' machines.

TYPE 1 :- Problems with n Jobs through two machines.

The algorithm, which is used to optimize the total elapsed time for processing 'n' jobs through two machines is called '**Johnson's algorithm**' & has the following steps.

Machine/Job	1	2	3	...	n
A	A_1	A_2	A_3	...	A_n
B	B_1	B_2	B_3	...	B_n

The problem is to sequence the jobs so as to minimize the total elapsed time.

Step 1:- Select the least processing time occurring in the list $A_1, A_2 \ldots A_n$ & $B_1, B_2, \ldots B_n$. Let this minimum processing time occurs for a job K.

Step 2:- If the shortest processing is for machine A, process the K^{th} job first & place it in the beginning of the sequence. If it is for machine B, process the K^{th} job last & place it at the end of the sequence.

Step 3:- When there is a tie in selecting the minimum processing time, then there may be three solutions,

(i) If the equal minimum values occur only for machine A select the job with larger processing time in B to be placed first in the job sequence.
(ii) If the equal minimum values occur only for machine B, select the job with larger processing time in A to be placed last in the job sequence.
(iii) If there are equal minimum values, one for each machine, then place the job in machine A first & the one in machine B last.

Step 4:- Delete the jobs already sequenced. If all the jobs have been sequenced, go to the next step. Otherwise, repeat steps 1 to 3.

Step 5:- In this step, determine the overall or total elapsed time and also the idle time on machines A & B as follows.

There are five jobs, each of which must go through the two machines A and B in the order AB. Processing times are given below.

Job	1	2	3	4	5
Machine A	5	1	9	3	10
Machine B	2	6	7	8	4
	x	x	x	x	x

The Optimal sequence for the task is

2	4	3	5	1

machine A $\rightarrow$ $\leftarrow$ machine B

Computation of total elapsed time and Idle time for Machine A and Machine B.

Jobs	Machine A			Machine B			Idle Time for machine B
	Time In	D	Time Out	Time in	D	Time out	
2	0	1	1	1	6	7	1
4	1	3	4	7	8	15	0
3	4	9	13	15	7	22	0
5	13	10	23	23	4	27	1
1	23	5	$\boxed{28}$	28	2	$\boxed{30}$	1
							3 hrs

Idle time for machine A = 30 - 28
= 2 hrs.

2. Find the sequence that minimizes the total elapsed time (in hours) required to complete the following tasks on two machines.

Task	A	B	C	D	E	F	G	H	I
Machine I	2	5	4	9	6	8	7	5	4
Machine II	6	8	7	4	3	9	3	8	11

Solⁿ: The optimal sequence for the task is,

A	I	C	B	H	F	D	E	G

Machine 1 →   ← Machine 2

The total elapsed time & idle time for both the machines are calculated.

Job	Machine A			Machine B			Idle time	
	In	D	Out	In	D	Out	M_1	M_2
A	0	2	2	2	6	8	0	2
I	2	4	6	8	11	19	0	0
C	6	4	10	19	7	26	0	0
B	10	5	15	26	8	34	0	0
H	15	5	20	34	8	42	0	0
F	20	8	28	42	9	51	0	0
D	28	9	37	51	4	55	0	0
E	37	6	43	55	3	58	0	0
G	43	7	50	58	3	61	61-50	0
							= 11 hrs	2 hrs.

Total time elapsed = 61 hours
Idle time for Machine 1 = 11 hrs
Idle time for machine 2 = 2 hrs.

Q. A company has six jobs, A to F. All the jobs have to go through two machine M1 and M2. The time required for the jobs on each machine in hours is given below. Find the optimum sequence that minimizes the total elapsed time.

Job	A	B	C	D	E	F
Machine 1	1	4	6	3	5	2
Machine 2	3	6	8	8	1	5

The optimum sequence for the task is,

A	F	D	B	C	E

$\longrightarrow$ Machine M_1 $\longleftarrow$ Machine M_2

The total elapsed time and idle time for each machine can be obtained as follows:-

Job	Machine A			Machine B			Idle time	
	In	D	Out	In	D	Out	M_1	M_2
A	0	1	1	1	3	4	0	1
F	1	2	3	4	5	9	0	0
D	3	3	6	9	8	17	0	0
B	6	4	10	17	6	23	0	0
C	10	6	16	23	8	31	0	0
E	16	5	21	31	1	$\boxed{32}$	0	0
							32-21 = 11hrs	1hr.

Total elapsed time = 32 hrs.
Idle time for Machine I = 11 hrs
Idle time for machine II = 1hr.

Unit-3 : Sequencing Problems. (continued)

Type II : Processing 'n' jobs through 3 machines A, B, C

Consider n jobs processing on three machines A, B, C in the order ABC.

The optimal sequence can be obtained by converting the problem into a two-machine problem.

From this, we get the optimum sequence using Johnson's algorithm.

- The smallest processing time on machine A is greater than or equal to the greatest processing time on machine B, i.e.,

$$\text{Min.}(A_i) \geq \text{Max.}(B_i)$$

- The smallest processing time on machine C is greater than or equal to the greatest processing time on machine B, i.e.,

$$\text{Max.}(B_i) \leq \text{Min.}(C_i)$$

(Atleast one of the above two conditions must be satisfied).

- If either or both of the above conditions are satisfied, then we replace the three machines by two fictitious machines G and H with corresponding processing times given by,

$$G_i = A_i + B_i$$
$$H_i = B_i + C_i$$

Where G_i & H_i are the processing times for i^{th} job on machine G and H respectively.

After calculating the new processing times, we determine optimal sequence of jobs for the machines G & H in the usual manner.

Q. A machine operator has to perform three operations, turning, threading and knurling, on a number of different jobs. The time required to perform these operations (in minutes) on each job is known. Determine the order in which the jobs should be processed in order to minimize the total time required to turn out all the jobs. Also find the minimum elapsed time

Job	1	2	3	4	5	6
Turning (A)	3	12	5	2	9	11
Threading (B)	8	6	4	6	3	1
Knurling (C)	13	14	9	12	8	13

Sol:- $\min(A_i, C_i) = (2, 8)$

$\max(B_i) = 8$

$\min(A_i) = 2 \not\geq \max(B_i) = 8$

$\min(C_i) = 8 \geq \max(B_i) = 8$ is satisfied.

We consider two fictitious machines G and H.

Job	1	2	3	4	5	6
G	11	18	9	8	12	12
H	21	20	13	18	11	14.

The Optimal sequence for the jobs.

4	3	1	6	2	5

$G \rightarrow$ $\leftarrow H$

To find the min. total elapsed time and idle time for machines A, B and C.

Job	Machine A			Machine B			Machine C		
	In	D	Out	In	D	Out	In	D	Out
4	0	2	2	2	6	8	8	12	20
3	2	5	7	8	4	12	20	9	29
1	7	3	10	12	8	20	29	13	42
6	10	11	21	21	1	22	42	13	55
2	21	12	33	33	6	39	55	14	69
5	33	9	42	42	3	45	69	8	$\boxed{77}$

Total elapsed time = 77 minutes.

Idle time for machine A = 77 - 42 = 35 minutes

Idle time for machine B = 2+1+11+3+(77-45)

= 49 minutes.

Idle time for machine C = 8 minutes.

Q. We have five jobs, each of which must go through the machines A, B and C in the order ABC. Determine the sequence that will minimize the total elapsed time.

Job No.	1	2	3	4	5
Machine A	5	7	6	9	5
Machine B	2	1	4	5	3
Machine C	3	7	5	6	7

Sol:- $Min(A_i) = 5$

$Min(C_i) = 3$

$Max(B_i) = 5$

$Min(A_i) \geq Max(B_i)$ is satisfied.

We consider two ficitious machines G and H.

Job No.	1	2	3	4	5
G	7	8	10	14	8
H	5	8	9	11	10

The Optimal sequence for jobs is.

5	2	4	3	1

g→ ←11

To find total elapsed time and idle time on three machines.

Job	Machine A			Machine B			Machine C			Idle time		
	In	D	Out	In	D	Out	In	D	Out	A	B	C
5	0	5	5	5	3	8	8	7	15	0	5	8
2	5	7	12	12	1	13	15	7	22	0	4	–
4	12	9	21	21	5	26	26	6	32	0	8	4
3	21	6	27	27	4	31	32	5	37	0	1	–
1	27	5	32	32	2	34	37	3	$\boxed{40}$	0	1	–
										40 − 32 = 8hrs	40 − 34 = 6 = 6 + 19 = 25hrs	= 12hrs

Total elapsed time = 40 hours.

Idle time for machine A = 8 hours.

Idle time for machine B = 25 hours.

Idle time for machine C = 12 hours.

Type III :- Problems with 'n' jobs and 'k' machines

Consider 'n' jobs (1, 2, ... n) processing through k machines $M_1, M_2, \ldots M_k$ in the same order. The iteration procedure of obtaining an optimal sequence is as follows.

Step 1 :- Find Min. M_{i1} and Min. M_{ik} and Max. of each of $M_{i2}, M_{i3}, \ldots, M_{ik-1}$ for $i = 1, 2, \ldots, n$

Step 2 :- Check whether

$$\text{Min } M_{i1} \geq \text{Max } M_{ij}, \text{ for } j = 2, 3, \ldots k-1 \text{ or}$$
$$\text{Min } M_{ik} \geq \text{Max } M_{ij}, \text{ for } j = 2, 3, \ldots k-1$$

Step 3 :- If the inequalities in step 2 are not satisfied, the method fails, otherwise, go to the next step.

Step 4 :- In addition to step 2, if $M_{i2} + M_{i3} + \ldots + M_{ik-1} = C$, where C is a positive fixed constant for all, $i = 1, 2, \ldots n$.

Then determine the optimal sequence for n jobs, where the two machines are M_1 and M_k in the order $M_1 M_k$, by using the optimum sequence algorithm.

Step 5 :- If the condition $M_{i2} + M_{i3} + \ldots + M_{ik-1} \neq C$ for all $i = 1, 2, \ldots n$, we define two machines G and H such that,

$$G_i = M_{i1} + M_{i2} + \ldots + M_{ik-1}$$
$$H_i = M_{i2} + M_{i3} + \ldots + M_{ik} \quad i = 1, 2 \ldots n$$

Determine the optimal sequence of performance of all jobs on G and H using the Optimum sequence algorithm for two machines.

Q. Four jobs 1, 2, 3 and 4 are to be processed on each of the five machines A, B, C, D and E in the order ABCDE. Find the total minimum elapsed time if no passing of jobs is permitted. Also find the idle time for each time.

Machines	Jobs			
	1	2	3	4
A	7	6	5	8
B	5	6	4	3
C	2	4	5	3
D	3	5	6	2
E	9	10	8	6

Sol:- $\text{Min}(A_i) = 5$

$\text{Max}(B_i, C_i, D_i) = (6, 5, 6)$

$\text{Min}(E_i) = 6$

$\Rightarrow \text{Min}(E_i) \geq \text{Max}(B_i, C_i, D_i)$ is satisfied.

$\therefore$ We can convert the problem into a two-machine problem.

Since $B_i + C_i + D_i \neq C$, (C - fixed constant)

We Define two machines G & H such that,

$G_i = A_i + B_i + C_i + D_i$

$H_i = B_i + C_i + D_i + E_i$, $i = 1, 2, 3, 4$.

Job	1	2	3	4
G	17	21	20	16
H	19	25	23	14

The Optimal sequence for this job is,

1	3	2	4

Job	Machine A			Machine B			Machine C			Machine D			Machine E			Idle time				
	In	D	Out	In	D	Out	In	D	Out	In	D	Out	In	D	Out	A	B	C	D	E
1	0	7	7	7	5	12	12	2	14	14	3	17	17	9	26	-	7	12	14	17
3	7	5	12	12	4	16	16	5	21	21	6	27	27	8	35	-	-	2	4	1
2	12	6	18	18	6	24	24	4	28	28	5	33	35	10	45	-	2	3	1	-
4	18	8	26	26	3	29	29	3	32	33	2	35	45	6	$\boxed{51}$	-	2	1	-	-
																51-26 = 25 hrs	51-29 = 22 +11 = 33 hrs	51-32 = 19 +18 = 37 hrs	51-35 = 16 +19 = 35 hrs	18 hrs

Total elapsed time = 51 hours

Idle time for machine A = 25 hrs

Idle time for machine B = 33 hrs.

Idle time for machine C = 37 hrs

Idle time for machine D = 35 hrs

Idle time for machine E = 18 hrs.

Type IV :- Problems with 2 Jobs through K machines

Consider two jobs, each of which is to be processed on K machines $M_1, M_2, \ldots, M_K$ in two different orders. The ordering of each of the two jobs through K machines is known in advance.

Each machine can perform only one job at a time. The objective is to determine the optimal sequence of processing the jobs so as to minimize the total elapsed time.

The Optimal sequence in this case can be obtained by making use of the graph.

The Procedure is given in the following steps:-

Step 1:- First draw a set of axes, where the horizontal axis represents processing time on job 1 & the vertical axis represents processing time on job 2.

Step 2:- Mark the processing time for job 1 and job 2 on the horizontal & vertical lines respectively, according to the given order of machines.

Step 3:- Construct various blocks starting from the origin, by pairing the same machines until the end point.

Step 4:- Draw the line starting from the origin to the end point by moving horizontally, vertically and diagonally along a line which makes an angle of 45° with the horizontal line. The horizontal segment of this line indicates that the first job is under process while second job is idle. Similarly, the vertical line indicates that the second job is under process while first job is idle. The diagonal segment of the line shows that the jobs are under process simultaneously.

Step5:- An Optimum path is the one that minimizes the idle time for both the jobs. Thus, we must choose the path on which diagonal movement is maximum.

Step6:- The total elapsed time is obtained by adding the idle time for either job to the processing time for that job.

Q. Use graphical method to minimize the time needed to process the following jobs on the machines shown below, i.e., for each machine find the job that should be done first. Also calculate the total time needed to complete both the jobs.

Job 1	Sequence of machine time	A 2	B 3	C 4	D 6	E 2
Job 2	Sequence of machine time	C 4	A 5	D 3	E 2	B 6

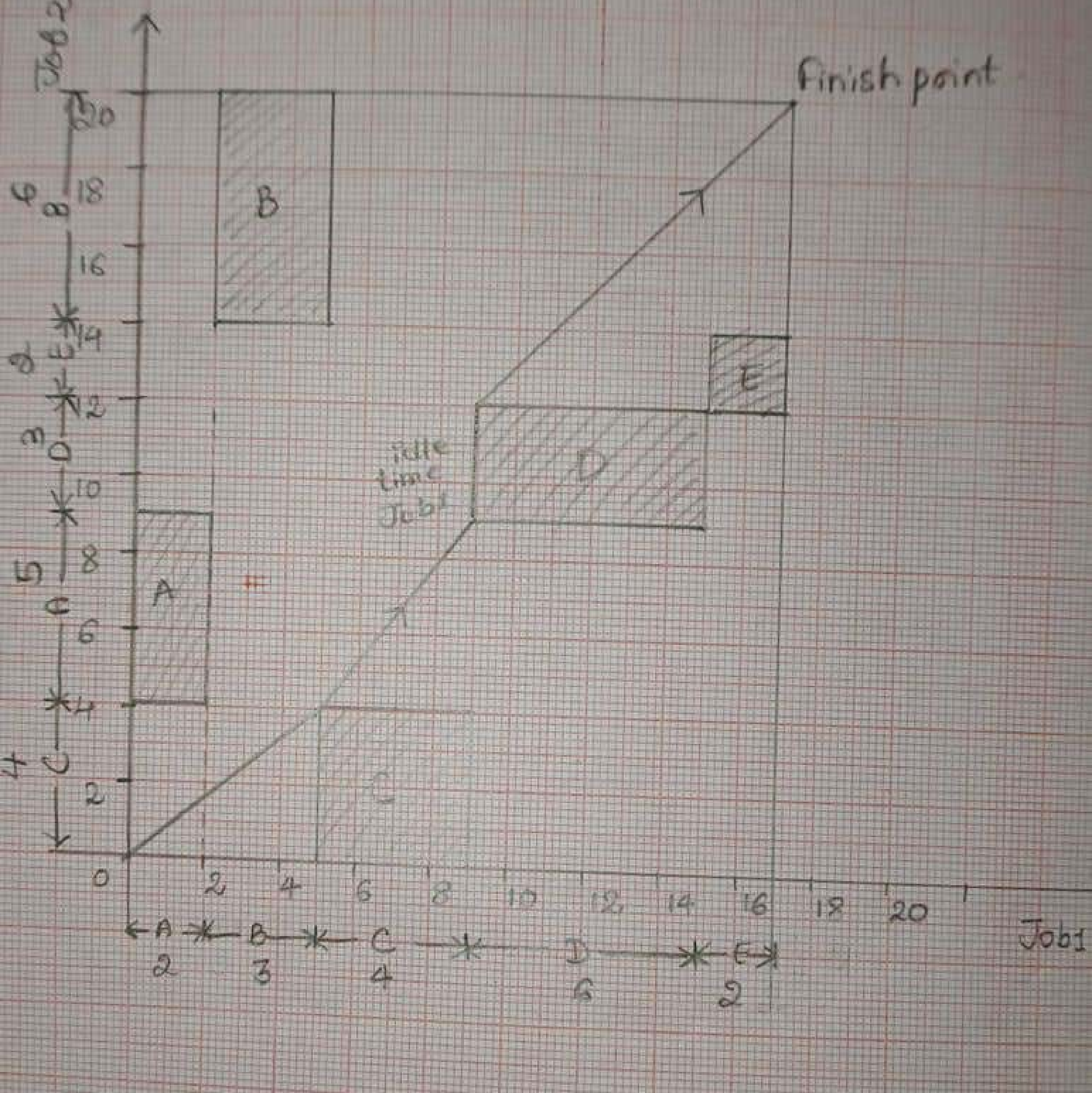
Sol:- An Optimum path minimizes the idle time for both the jobs.

Idle time of job 1 = 3 hours.

Idle time of job 2 = 0 hours.

Elapsed time, job 1 = 17+3 = 20 hours

Elapsed time, job 2 = 20+0 = 20 hours.



Job 2
Finish point
B
A
C
D
E
idle time Job1
Job1
0
2
4
6
8
10
12
14
16
18
20
A 2
B 3
C 4
D 6
E 2
C 4
A 5
D 3
E 2
B 6

Q.

Job 1	Sequence of machine Time	A B C D E 3 4 2 6 2
Job 2	Sequence of machine Time	B C A D E 5 4 3 2 6.

Sol:-

In this problem the idle time for the chosen path is 5 hours for job 1 & 2 hours for Job 2. Therefore, the total elapsed time is obtained as follows.

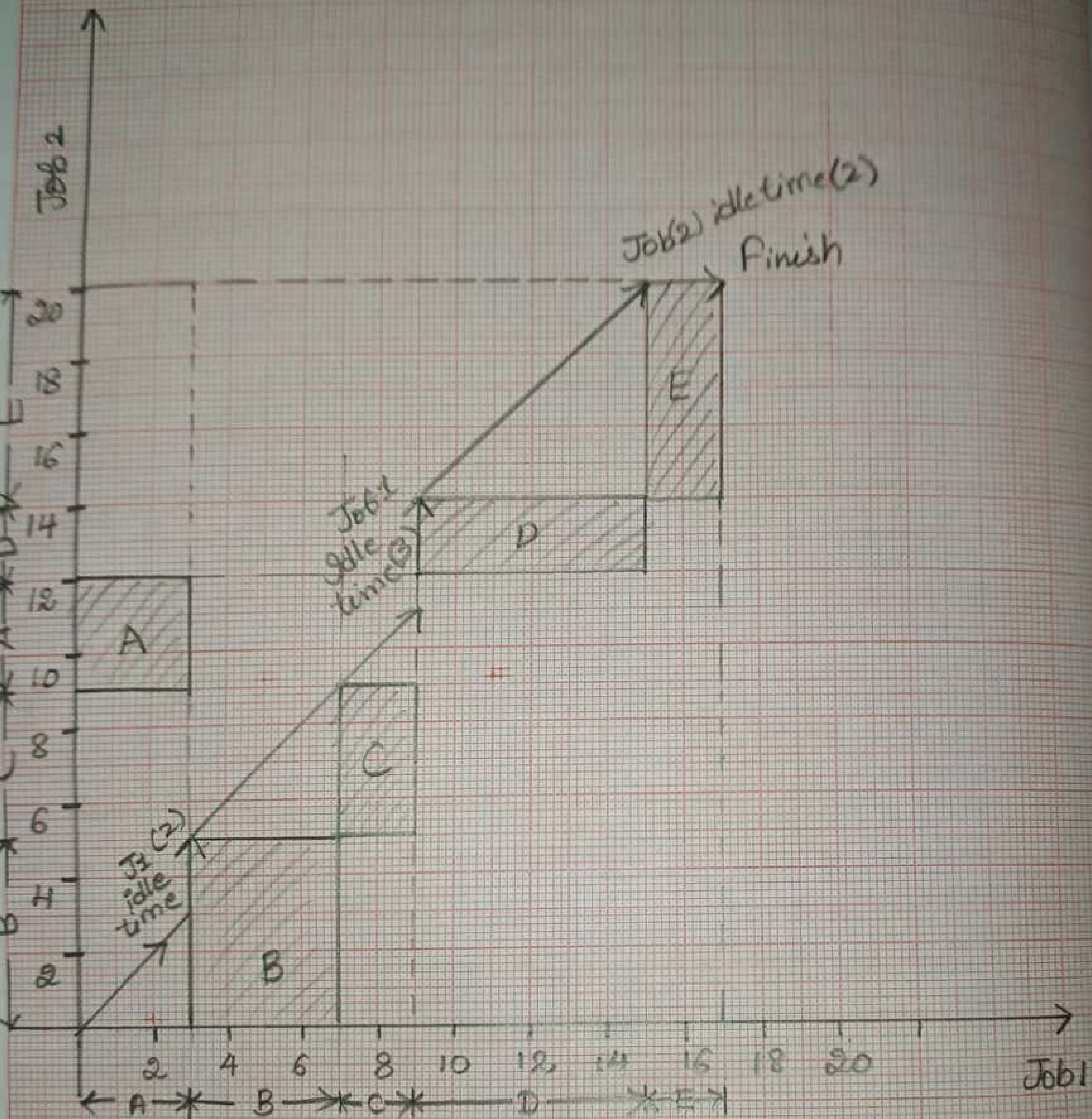
Elapsed time, job 1 =
Processing time for job 1 + idle time for job 1

$= 17 + (2+3) = 22$ hours.

Elapsed time, job 2 =
Processing time for job 2 + idle time for job 2

$= 20 + 2 = 22$ hours.

2.


Job 2
Job(2) idle time(2)
Finish
20
18
16
14
12
10
8
6
4
2
E
D
A
C
B
Job 1 idle time(3)
J1 (2) idle time
2
4
6
8
10
12
14
16
18
20
Job1
A
B
C
D
E