Unit - 2

TRANSPORTATION PROBLEM.

A transportation problem in OR is a special type of LPP used to minimize the transportation cost and allocate resources from 'm' sources to 'n' destinations.

The main objective of transportation problem is to deliver the resources at the minimum cost.

Formulation of Transportation Problem.

Here we are supplying the resources from 'm' source (S_i) to 'n' destinations (D_j) such that:

a_i - the quantity available at the source S_i.

b_j - the quantity required at the destination D_j.

C_{ij} - cost of transportation of one unit resource from S_i to D_j

x_{ij} - Units of resources transported from S_i to D_j.

$1 \le i \le m$, $1 \le j \le n$

The total transportation cost is

$$= (C_{11}x_{11} + C_{12}x_{12} + C_{13}x_{13}) + (C_{21}x_{21} + C_{22}x_{22} + \ldots + C_{2n}x_{2n}) + \ldots + (C_{m1}x_{m1} + C_{m2}x_{m2} + \ldots + C_{mn}x_{mn}).$$

Standard form:-

Destination

Source	D_1	D_2	D_3	...	D_n	Supply
S_1	C_{11}	C_{12}	C_{13}	...	C_{1n}	a_1
S_2	C_{21}	C_{22}	C_{23}	...	C_{2n}	a_2
S_3	C_{31}	C_{32}	C_{33}	...	C_{3n}	a_3
⋮	⋮	⋮	⋮		⋮	⋮
S_m	C_{m1}	C_{m2}	C_{m3}	...	C_{mn}	a_m
Demand	b_1	b_2	b_3	...	b_n	

Definitions related to transportation problem.

1. The feasible solution to a transportation problem is def as a solution that satisfies all the constraints of a problem.

2. Basic feasible solution :- A feasible solution is called a bas feasible solution if it contains not more than m+n-1 allocations, where m is the number of rows & n is the number of columns in a transportation problem.

3. Optimal solution :- Optimal solution is a feasible solution which optimizes the total transportation cost.

4. Non-degenerate basic feasible solution : If a basic feasible solution to a transportation problem contains exactly m+n-1 allocations in independent positions, it is called a Non-degenerate basic feasible solution.

5. Degeneracy :- If a basic feasible solution to a transportation problem contains less than m+n-1 allocations, it is called a degenerate basic feasible solution.

Types of Transportation Problems.

Balanced Transportation problem.

Total Supply = Total Demand i.e. $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Unbalanced Transportation problem.

Total Supply ≠ Total Demand i.e.,

- $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, a dummy destination is added in the matrix with zero cost elements.
- $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$, a dummy source is included in the matrix with zero cost elements.

METHODS OF FINDING INITIAL BASIC FEASIBLE SOLUTION.

In order to find the IBFS, total supply must be equal to the total demand.

1. North West Corner Method
2. Least Cost Method or Matrix minimum method.
3. Vogel's Approximation Method.

North West Corner Method

NWC Rule :-

Step 1: Select the upper left corner cell of the transportation matrix and allocate $\min(s_i, d_j)$

Step 2: a) Subtract this value from supply and demand of respective row & column.

b). If the supply is 0, then cross (strike) that row & move down to the next cell.

c). If the demand is 0, then cross (strike) that column & move right to the next cell.

d). If supply & demand both are 0, then strike both row & column & move diagonally to the next cell.

Step-3 :- Repeat this steps until all supply & demand values are 0.

Q1. Find solution using North-West Corner method.

	D1	D2	D3	D4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	

Sol:- Total no. of supply constraints = 3
Total no. of demand constraints = 4

Problem table is,

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	

The rim values for $S_1 = 7$ and $D_1 = 5$ are compared. i.e. min(7,5)=5 is assigned to S_1D_1. This meets the complete demand of D_1 and leaves $7-5=2$ units with S_1.

Table 1:-

	D_1	D_2	D_3	D_4	Supply
S_1	19(5)	30	50	10	2
S_2	70	30	40	60	9
S_3	40	8	70	20	18.
Demand	0	8	7	14	

Move horizontally,
The rim values for $S_1=2$ & $D_2=8$ are compared
The smaller of the two i.e. min (2,8) = 2 is assigned to S_1D_2.
This exhausts the capacity of S_1 and leaves $8-2=6$ units with D_2.

Table 2:-

	D_1	D_2	D_3	D_4	Supply
S_1	19(5)	30(2)	50	10	0
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	0	6	7	14	

Move vertically,
The rim values for $S_2=9$ and $D_2=6$ are compared.
The smaller of the two, i.e., min (9,6) = 6 is assigned to S_2D_2.
This meets the complete demand of D_2 & leaves $9-6=3$ units with S_2.

Table 3:-

	D_1	D_2	D_3	D_4	Supply
S_1	19 (5)	30 (2)	50	10	0
S_2	70	30 (6)	40	60	3
S_3	40	8	70	20	18
Demand	0	0	7	14	

Move horizontally, The rim values for $S_2 = 3$ & $D_3 = 7$ are compared. $\min(3,7) = 3$ is assigned to S_2D_3. This exhausts the capacity of S_2 & leaves $7-3=4$ units with D_3.

Table-4:-

	D_1	D_2	D_3	D_4	Supply
S_1	19 (5)	30 (2)	50	10	0
S_2	70	30 (6)	40 (3)	60	0
S_3	40	8	70	20	18
Demand	0	0	4	14	

Move vertically, The rim values for $S_3 = 18$ & $D_3 = 4$ are compared. $\min(18,4) = 4$ is assigned to S_3D_3. This meets the complete demand of D_3 & leaves $18-4=14$ units with S_3.

Table-5:-

	D_1	D_2	D_3	D_4	Supply
S_1	19 (5)	30 (2)	50	10	0
S_2	70	30 (6)	40 (3)	60	0
S_3	40	8	70 (4)	20	14
Demand	0	0	0	14	

Move horizontally, The rim values for $S_3 = 14$ & $D_4 = 14$ are compared. $\min(14,14) = 14$ is assigned to S_3D_4.

Table-6:-

	D_1	D_2	D_3	D_4	Supply
S_1	19 (5)	30 (2)	50	10	0
S_2	70	30 (6)	40 (3)	60	0
S_3	40	8	70 (4)	20 (14)	0
Demand	0	0	0	0	

The minimum transportation cost $= 19\times5 + 30\times2 + 30\times6 + 40\times3 + 70\times4 + 20\times14 = 1015$

Here the number of allocated cells $= 6$ is equal to $m+n-1 = 6$.

This solution is non-degenerate.

LEAST COST METHOD (LCM) (OR) MATRIX MINIMA METHOD.

Rule :-

Step 1:- Select the cell having minimum unit cost c_{ij} and allocate as much as possible, i.e., min (s_i, d_j)

Step 2:- a) Subtract this min value from supply s_i & demand d_j

b) If the supply s_i is 0, then cross that row & if the demand d_j is 0, then cross the column.

c) If min unit cost cell is not unique, then select the cell where maximum allocation can be possible.

Step 3:- Repeat this steps for all uncrossed rows & columns until all supply & demand values are zero.

Q. Obtain an initial basic feasible solution to the following TP using the matrix minima method.

	D_1	D_2	D_3	D_4	Supply
S_1	1	2	3	4	6
S_2	4	3	2	0	8
S_3	0	2	2	1	10
Demand	4	6	8	6	

Soln:- Total no. of supply constraints = 24
Total no. of demand constraints = 24.

	D_1	D_2	D_3	D_4	Supply
S_1	1	2 (6)	3	4	~~6~~ 0
S_2	4	3	2 (2)	0 (6)	~~8~~ ~~2~~ 0
S_3	0 (4)	2	2 (6)	1	~~10~~ ~~6~~ 0
Demand	~~4~~ 0	~~6~~ 0	~~8~~ ~~2~~ 0	~~6~~ 0	

The minimum transportation cost $= 2\times6 + 2\times2 + 0\times6 + 0\times4 + 2\times6 = 12+4+12 = 28.$

Here, the no. of allocated cells $= 5$ is not equal to $m+n-1 = 3+4-1 = 6.$

Hence, the solution is degenerate.

Find the non-degenerate basic feasible solution for the following transportation problem by least cost method.

	Destination				Supply
Source	10	20	5	7	10
	13	9	12	8	20
	4	5	7	9	30
	14	7	1	0	40
	3	12	5	19	50
Demand	60	60	20	10	

Total no. of Supply constraints $= 5$

Total no. of demand constraints $= 4$

	D_1	D_2	D_3	D_4	Supply
S_1	10	20 (10)	5	7	~~10~~ 0
S_2	13	9 (20)	12	8	~~20~~ 0
S_3	4 (10)	5 (20)	7	9	~~30~~ ~~20~~ 0
S_4	14	7 (10)	1 (20)	0 (10)	~~40~~ ~~30~~ ~~10~~ 0
S_5	3 (50)	12	5	19	~~50~~ 0
	~~60~~ ~~10~~ 0	~~60~~ ~~40~~ ~~30~~ ~~10~~ 0	~~20~~ 0	~~10~~ 0	

Minimum Transportation cost $=$ Rs. $20\times10 + 9\times20 + 4\times10 + 0\times20 + 7\times10 + 1\times20 + 0\times10 + 3\times50$

$=$ Rs. 760/-

Here, no. of allocated cells $= 8$

$(m+n-1) = (5+4-1) = 8$ ∴ The sol^n is non-degenerate.

Vogel's Approximation Method (VAM) or Penalty method.

This method is preferred over the NWCM and LCM, because the IBFS obtained by this method is either optimal solution or very nearer to the optimal solution.

Steps:

Step 1:- Find the cells having the smallest and next to smallest cost in each row and write the difference (called penalty) along the side of the table in row penalty.

Step 2:- Find the cells having the smallest and next to smallest cost in each column and write the difference along the side of the table in each column penalty.

Step 3:- Select the row or column with the maximum penalty & find cell that has least cost in selected row or column. Allocate as much as possible in this cell.
(If their is a tie in the values of penalties then select the cell where maximum allocation can be possible).

Step 4:- Adjust the supply & demand and cross out the satisfied row or column.

Step 5:- Repeat this steps until all supply & demand values are 0.

Q 1. Find solution using Vogel's Approximation method.

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	1
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	

Sol:- No. of Supply constraints = 3

No. of Demand constraints = 4

	D_1	D_2	D_3	D_4	Supply	Row penalty				
S_1	~~19~~ (5)	~~30~~	~~50~~	~~10~~ (2)	~~7~~ ~~2~~ 0	9	9	40	40	–
S_2	~~70~~	~~30~~	~~40~~ (7)	~~60~~ (2)	~~9~~ ~~7~~ 0	10	20	20	20	20
S_3	~~40~~	~~8~~ (8)	~~70~~	~~20~~ (10)	~~18~~ ~~10~~ 0	12	20	$\boxed{50}$	–	–
Demand	~~5~~ 0	~~8~~ 0	~~7~~ 0	~~14~~ ~~12~~ ~~2~~ 0						
Column	21	$\boxed{22}$	10	10						
Penalty	$\boxed{21}$	–	10	10						
	–	–	10	10						
	–	–	30	40						
	–	–	10	$\boxed{50}$						
	–	–	40	$\boxed{60}$						
	–	–	$\boxed{40}$	–						
	–	–	–	–						

The minimum total transportation cost,

$= 19\times5 + 10\times2 + 40\times7 + 60\times2 + 8\times8 + 20\times10$

$= 779$ Rs.

Here, the no. of allocated cells = 6 is equal to $m+n-1$

$= 3+4-1 = 6$

$\therefore$ The solⁿ is non-degenerate.

Q2. Use Vogel's approximation method to obtain an IBFS of the given transportation problem.

		D_1	D_2	D_3	D_4	Supply
	S_1	3	3	4	1	100
Source	S_2	4	2	4	2	125
	S_3	1	5	3	2	75
Demand		120	80	75	25	

Soln:- No. of supply constraints = 3
No. of demand constraints = 4
Total supply = Total demand.

	D_1	D_2	D_3	D_4	Supply	Row penalty
S_1	3 (45)	3	4 (30)	1 (25)	~~100~~ ~~75~~ ~~30~~ 0	2 [2] 0 [1] 4 0
S_2	4	2 (80)	4 (45)	2	~~125~~ ~~45~~ 0	0 0 [2] 0 4 –
S_3	1 (75)	5	3	2	~~75~~ 0	1 – – – – –
Demand	~~120~~ ~~45~~ 0	~~80~~ 0	~~75~~ ~~30~~ 0	~~25~~ 0		

Column penalty

D_1	D_2	D_3	D_4
[2]	1	0	1
1	1	0	1
1	1	0	–
1	–	0	–
–	–	0	–
–	–	–	–

The minimum transportation cost

$= 3\times45 + 4\times30 + 1\times25 + 2\times80 + 4\times45 + 1\times75$

$= 695$

Here, the no. of allocated cells = 6 is equal to $m+n-1 = 3+4-1 = 6$

$\therefore$ This soln is non-degenerate.

Q3. Find the IBFS for the transportation problem by VAM.

	D_1	D_2	D_3	D_4	Supply
S_1	11	13	17	14	250
S_2	16	18	14	10	300
S_3	21	24	13	10	400
Demand	200	225	275	250	

Sol:- No. of Supply constraints = 3
No. of Demand constraints = 4
Total supply = Total Demand.

	D_1	D_2	D_3	D_4	Supply	Row penalty					
S_1	~~11~~ 200	~~13~~ 50	~~17~~	~~14~~	~~250~~ ~~50~~ 0	2	1	–	–	–	–
S_2	~~16~~	~~18~~ 175	~~14~~	~~10~~ 125	~~300~~ ~~125~~ 0	4	4	4	$\boxed{4}$	–	–
S_3	~~21~~	~~24~~	~~13~~ 275	~~10~~ 125	~~400~~ ~~125~~ 0	3	3	3	3	3	10
Demand	~~200~~ 0	~~225~~ ~~175~~ 0	~~275~~ 0	~~250~~ ~~125~~ 0							
Column penalty	$\boxed{5}$	5	1	0							
	–	$\boxed{5}$	1	0							
	–	$\boxed{6}$	1	0							
	–	–	1	0							
	–	–	$\boxed{13}$	10							
	–	–	–	10							

The minimum transportation cost

$= 11\times 200 + 13\times 50 + 18\times 175 + 10\times 125 + 13\times 275 + 10\times 125$

$=$ Rs. 12,075.

Here, the no. of allocated cells = 6 is equals $m+n-1 = 3+4-1 = 6$!

$\therefore$ The sol^n is non-degenerate.

UNBALANCED TRANSPORTATION PROBLEM.

If the given transportation problems is unbalanced one, i.e., if $\sum a_i \neq \sum b_j$. then convert this into a balanced one by introducing a dummy source or dummy destination with few cost.

Ex:- Solve the transportation problem.

		A	B	C	D	Supply
	1	11	20	7	8	50
Source	2	21	16	20	12	40
	3	8	12	18	9	70
Demand		30	25	35	40	

Sol:- $\sum a_i$ = Total Supply = $50+40+70 = 160$
$\sum b_j$ = Total Demand = $30+25+35+40 = 130$.

$\sum a_i > \sum b_j$, the given problem is an unbalanced transportation problem.

To convert into a balanced one, we introduce a dummy destination with zero unit cost & having demand equal to $(160-130) = 30$ units.

Hence the given problem becomes,

	A	B	C	D	E	Supply
1	11	20	7	8	0	50
2	21	16	20	12	0	40
3	8	12	18	9	0	70
Demand	30	25	35	40	30	

By using VAM the initial solution is given by,

	A	B	C	D	E	Supply	Row penalty					
1	11	20	7 (35)	8 (15)	0	~~50~~ ~~15~~ 0	7	1	3	3	3	8
2	21	16	26	12 (10)	0 (30)	~~40~~ ~~10~~ 0	(12)	4	4	(9)	-	-
3	8 (30)	12 (25)	18	9 (15)	0	~~70~~ ~~45~~ ~~15~~ 0	8	1	1	1	(8)	(9)
Demand	~~30~~ 0	~~25~~ 0	~~35~~ 0	~~40~~ ~~30~~ ~~15~~ 0	~~30~~ 0							
Column Penalty	3	4	11	1	0							
	3	4	(11)	1	0							
	3	(4)	-	1	-							
	3	-	-	1	-							
	3	-	-	1	-							
	-	-	-	1	-							
	-	-	-	8	-							

The minimum transportation cost

$= 7\times35 + 8\times15 + 12\times10 + 0\times30 + 8\times30 + 12\times25 + 9\times15$

$=$ Rs. 1,160/-

Ex:- Solve the transportation problem,

		D_1	D_2	D_3	D_4	Supply
Source	S_1	6	1	9	3	70
	S_2	11	5	2	8	55
	S_3	10	12	4	7	70
	Demand	85	35	50	45	

Sol:- Σb_j = Total demand $= 85+35+50+45 = 215$

Σa_i = Total supply $= 70+55+70 = 195$.

$\Sigma a_i < \Sigma b_j$, the given problem is unbalanced transportation problem

To convert this into balanced transportation problem introduce a dummy source S_4 with zero unit transportation costs & having supply equal to $215 - 195 = 20$ units. The given problem becomes,

Destination

		D_1	D_2	D_3	D_4	Supply
Source	S_1	6	1	9	3	70
	S_2	11	5	2	8	55
	S_3	10	12	4	7	70
	S_4	0	0	0	0	20
	Demand	85	35	50	45	215

By Using Vogel's approximation method, the initial solⁿ is:-

	D_1	D_2	D_3	D_4	Supply	Row penalty						
S_1	6	~~1~~ (35)	~~9~~	~~3~~ (35)	~~70~~ ~~35~~ 0	2	2	3	3	-	1	-
S_2	~~11~~ (5)	~~5~~	~~2~~ (50)	~~8~~	~~55~~ ~~5~~ 0	3	3	6	3	3	1	-
S_3	~~10~~ (60)	~~12~~	~~4~~	~~7~~ (10)	~~70~~ ~~60~~ 0	3	3	3	3	3	10	10
S_4	~~0~~ (20)	~~0~~	~~0~~	~~0~~	~~20~~ 0	-	-	-	-	-	-	-
Demand	~~85~~ ~~65~~ ~~60~~ 0	~~35~~ 0	~~50~~ 0	~~45~~ ~~10~~ 0								

Column penalty	6	1	2	3
	4	4	2	4
	4	-	2	4
	4	-	-	4
	1	-	-	1
	1	-	-	-
	10	-	-	-

$\therefore$ Minimum transportation cost $= 1 \times 35 + 3 \times 35 + 11 \times 5 + 2 \times 50 + 10 \times 60 + 7 \times 10 + 0 \times 20 = \text{Rs. } 965/-$

The no. of allocated cells $= 7$ is equal to $m+n-1 = 4+4-1 = 7$

$\therefore$ This solⁿ is non-degenerate.

TRANSPORTATION ALGORITHM [OR] MODI METHOD

Modified Distribution method – Test for Optimal solution

Step 1: Find an initial basic feasible solution using any one of the three methods NWCM, LCM or VAM

Step 2: Find u_i and v_j for rows and columns. To start

a). Assign 0 to u_i or v_j where maximum number of allocation in a row or column respectively.

b). Calculate other u_i's & v_j's using $C_{ij} = u_i + v_j$, for all occupied cells

Step 3: For all unoccupied cells, calculate $d_{ij} = C_{ij} - (u_i + v_j)$.

Step 4: Check the sign of d_{ij}

a). If $d_{ij} \geq 0$, then current basic feasible solution is optimal & stop this procedure.

b). If atleast one $d_{ij} = 0$ then alternate solution exists, with different set allocation & same transportation cost. Now stop this procedure

c) If $d_{ij} < 0$, then the given solution is not an optimal solution & further improvement in the solution is possible

Step 5: Select the unoccupied cell with the largest negative value of d_{ij}, and included in the next solution.

Step 6: Draw a closed path (or loop) from the unoccupied cell (selected in the previous step). The right angle turn in this path is allowed only at occupied cells & at the original unoccupied cell. Mark (+) and (–) sign alternatively at each corner, starting from the original unoccupied cell.

Step 7: 1) Select the minimum value from cells marked with (–) sign of the closed path.

2) Assign this value to selected unoccupied cells (So unoccupied cell becomes occupied)

3) Add this value to the other occupied cells marked with (+) sign.

4) Subtract this value to the other occupied cells marked with (−) sign.

Step 8:- Repeat step (2) to step (7) until optimal solution is obtained. This procedure stops when all $d_{ij} \geq 0$ for unoccupied cells.

Q. Find solution using Vogel's approximation method, also find optimal solution using modi method.

Destination

		D1	D2	D3	D4	Supply
	S1	21	16	25	13	11
Source	S2	17	18	14	23	13
	S3	32	27	18	41	19
	Demand	6	10	12	15	

Sol:- Total number of supply constraints = 3

Total number of demand constraints = 4

Total supply = Total demand

∴ The transportation problem is balanced.

	D1	D2	D3	D4	Supply	Row penalty
S1	~~21~~	~~16~~	~~25~~	~~13~~ (11)	~~11~~ 0	3 – – – – –
S2	~~17~~ (6)	~~18~~	~~14~~	~~23~~ (4)	~~13~~ 0	3 3 3 4 18 18
S3	~~32~~	27 (7)	~~18~~ (12)	~~41~~	~~19~~ 0	9 9 9 9 27 –
Demand	~~6~~ 0	~~10~~ 0	~~12~~ 0	~~15~~ 0		
Column	4	2	4	10		
penalty	15	9	4	18		
	15	9	4	–		
	–	9	4	–		
	–	9	–	–		
	–	18	–	–		

The minimum total transportation cost
$= 13\times11 + 17\times6 + 18\times3 + 23\times4 + 27\times7 + 18\times12 = 796/-$

Here, the number of allocated cells $= 6$ is equal to $m+n-1 = 3+4-1 = 6$

∴ This solution is non-degenerate.

Optimality test using MODI Method,

Allocation Table is,

	D_1	D_2	D_3	D_4	Supply
S_1	21	16	25	13 (11)	11
S_2	17 (6)	18 (3)	14	23 (4)	13
S_3	32	27 (7)	18 (12)	41	19
Demand	6	10	12	15	

Iteration-1

1. Find u_i and v_j for all occupied cells (i,j), where $C_{ij} = u_i + v_j$.

i) Substituting, $u_2 = 0$ (2^{nd} row contains maximum no. of allocations) we get,

ii) $C_{21} = u_2 + v_1 \Rightarrow v_1 = 17$

iii) $C_{22} = u_2 + v_2 \Rightarrow v_2 = 18$

iv) $C_{32} = u_3 + v_2 \Rightarrow u_3 = 9$

v) $C_{33} = u_3 + v_3 \Rightarrow v_3 = 9$

vi) $C_{24} = u_2 + v_4 \Rightarrow v_4 = 23$

vii) $C_{14} = u_1 + v_4 \Rightarrow u_1 = -10$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	21	16	25	13 (11)	11	$u_1 = -10$
S_2	17 (6)	18 (3)	14	23 (4)	13	$u_2 = 0$
S_3	32	27 (7)	18 (12)	41	19	$u_3 = 9$
Demand	6	10	12	15		
v_j	$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$		

2. Find d_{ij} for all unoccupied cells (i,j), where $d_{ij} = c_{ij} - (u_i + v_j)$

i) $d_{11} = c_{11} - (u_1 + v_1) = 21 - (-10 + 17) = 14$
ii) $d_{12} = c_{12} - (u_1 + v_2) = 16 - (-10 + 18) = 8$
iii) $d_{13} = c_{13} - (u_1 + v_3) = 25 - (-10 + 9) = 26$
iv) $d_{23} = c_{23} - (u_2 + v_3) = 14 - (0 + 9) = 5$
v) $d_{31} = c_{31} - (u_3 + v_1) = 32 - (9 + 17) = 6$
vi) $d_{34} = c_{34} - (u_3 + v_4) = 41 - (9 + 23) = 9$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	21(14)	16(8)	25(26)	13(11)	11	$u_1 = -10$
S_2	17(6)	18(3)	14(5)	23(4)	13	$u_2 = 0$
S_3	32(6)	27(7)	18(12)	41(9)	19	$u_3 = 9$
Demand	6	10	12	15		
v_j	$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$		

Since all $d_{ij} \geq 0$

So final optimal solution is arrived.

	D_1	D_2	D_3	D_4	Supply
S_1	21	16	25	13(11)	11
S_2	17(6)	18(3)	14	23(4)	13
S_3	32	27(7)	18(12)	41	19
Demand	6	10	12	15	

The minimum transportation cost

$= 13 \times 11 + 17 \times 6 + 18 \times 3 + 23 \times 4 + 27 \times 7 + 18 \times 12 =$ Rs. 796/-

Q. Find the optimal transportation cost of the following matrix using least cost method for finding the initial solution.

		Market					
		A	B	C	D	E	Available
Factory	P	4	1	2	6	9	100
	Q	6	4	3	5	7	120
	R	5	2	6	4	8	120
	Demand	40	50	70	90	90	

Solⁿ: Since $\Sigma a_i = \Sigma b_j = 340$, the given transportation problem is balanced.

∴ There exists a basic feasible solution to this problem.
By using least cost method, the initial solution is as shown in the following table.

	A	B	C	D	E	Available
P	~~4~~	~~1~~ (50)	~~2~~ (50)	~~6~~	~~9~~	~~100~~ ~~50~~ 0
Q	~~6~~ (10)	~~4~~	~~3~~ (20)	~~5~~	~~7~~ (90)	~~120~~ ~~100~~ ~~90~~ 0
R	~~5~~ (30)	~~2~~	~~6~~	~~4~~ (90)	~~8~~	~~120~~ ~~30~~ 0
Demand	~~40~~ ~~10~~ 0	~~50~~ 0	~~70~~ ~~20~~ 0	~~90~~ 0	~~90~~ 0	

Minimum transportation cost = Rs. $1 \times 50 + 2 \times 50 + 6 \times 10 + 3 \times 20 + 7 \times 90 + 5 \times 30 + 4 \times 90$
= Rs. 1,410.

Since, the no. of allocation is 7 is equal to $m+n-1 = 3+5-1$.
The solution is non-degenerate.

For Optimality:-
We apply MODI method.

(P.T.O)

Allocation table is,

	A	B	C	D	E	Supply
P	4	1 (50)	2 (50)	6	9	100
Q	6 (10)	4	3 (20)	5	7 (90)	120
R	5 (30)	2	6	4 (90)	8	120
Demand	40	50	70	90	90.	

Iteration - 1 :-

1. Find u_i & v_j for all occupied cells (i,j), where $C_{ij} = u_i + v_j$.

i) Substituting $u_2 = 0$.

ii) $C_{12} = u_1 + v_2 \Rightarrow 1 = -1 + v_2 \Rightarrow v_2 = 2$

iii) $C_{13} = u_1 + v_3 \Rightarrow 2 = u_1 + 3 \Rightarrow u_1 = -1$

iv) $C_{21} = u_2 + v_1 \Rightarrow v_1 = 6$

v) $C_{23} = u_2 + v_3 \Rightarrow v_3 = 3$

vi) $C_{25} = u_2 + v_5 \Rightarrow v_5 = 7$

vii) $C_{31} = u_3 + v_1 \Rightarrow 5 = u_3 + 6 \Rightarrow u_3 = -1$

viii) $C_{34} = u_3 + v_4 \Rightarrow 4 = -1 + v_4 \Rightarrow v_4 = 5$.

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	4	1 (50)	2 (50)	6	9	100	$u_1 = -1$
S_2	6 (10)	4	3 (20)	5	7 (90)	120	$u_2 = 0$
S_3	5 (30)	2	6	4 (90)	8	120	$u_3 = -1$
Demand	40	50	70	90	90		
v_j	$v_1 = 6$	$v_2 = 2$	$v_3 = 3$	$v_4 = 5$	$v_5 = 7$		

2. Find d_{ij} for all unoccupied cells (i,j), where $d_{ij} = c_{ij} - (u_i + v_j)$

1. $d_{11} = c_{11} - (u_1 + v_1) = 4 - (-1+6) = -1$
2. $d_{14} = c_{14} - (u_1 + v_4) = 6 - (-1+5) = 2$
3. $d_{15} = c_{15} - (u_1 + v_5) = 9 - (-1+7) = 3$
4. $d_{22} = c_{22} - (u_2 + v_2) = 4 - (0+2) = 2$

5. $d_{24} = C_{24} - (u_2 + v_4) = 5 - (0+5) = 0$

6. $d_{32} = C_{32} - (u_3 + v_2) = 2 - (-1+2) = 1$

7. $d_{33} = C_{33} - (u_3 + v_3) = 6 - (-1+3) = 4$

8. $d_{35} = C_{35} - (u_3 + v_5) = 8 - (-1+7) = 2$

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	4(-1)(+)	1(50)	2(50)(-)	6(2)	9(3)	100	$u_1 = -1$
S_2	6(10)(-)	4(2)	3(20)(+)	5(0)	7(90)	120	$u_2 = 0$
S_3	5(30)	2(1)	6(4)	4(90)	8(2)	120	$u_3 = -1$
Demand	40	50	70	90	90		
v_j	$v_1 = 6$	$v_2 = 2$	$v_3 = 3$	$v_4 = 5$	$v_5 = 7$		

Choose the minimum negative value from all d_{ij}, $d_{11} = -1$
closed path $S_1D_1 \to S_1D_3 \to S_2D_3 \to S_2D_1$

* Minimum allocated value among all negative position (-) on closed path = 10.

* Subtract 10 from all (-) and add to all (+)

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	4(10)	1(50)	2(40)	6	9	100
S_2	6	4	3(30)	5	7(90)	120
S_3	5(30)	2	6	4(90)	8	120
Demand	40	50	70	90	90	

Iteration-2: 1. Find u_i & v_j for all occupied cells (i,j), where $c_{ij} = u_i + v_j$.

1. Substituting, $u_1 = 0$, we get
2. $C_{11} = u_1 + v_1 \Rightarrow v_1 = C_{11} - u_1 \Rightarrow v_1 = 4 - 0 \Rightarrow v_1 = 4$
3. $C_{31} = u_3 + v_1 \Rightarrow u_3 = C_{31} - v_1 \Rightarrow u_3 = 5 - 4 \Rightarrow u_3 = 1$
4. $C_{34} = u_3 + v_4 \Rightarrow v_4 = C_{34} - u_3 \Rightarrow v_4 = 4 - 1 \Rightarrow v_4 = 3$
5. $C_{12} = u_1 + v_2 \Rightarrow v_2 = C_{12} - u_1 \Rightarrow v_2 = 1 - 0 \Rightarrow v_2 = 1$

6. $C_{13} = u_1 + v_3 \Rightarrow v_3 = C_{13} - u_1 \Rightarrow v_3 = 2$

7. $C_{23} = u_2 + v_3 \Rightarrow u_2 = C_{23} - v_3 \Rightarrow u_2 = 3 - 2 \Rightarrow u_2 = 1$

8. $C_{25} = u_2 + v_5 \Rightarrow v_5 = C_{25} - u_2 \Rightarrow v_5 = 7 - 1 \Rightarrow v_5 = 6$

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	4(10)	1(50)	2(40)	6(3)	9(3)	100	$u_1=0$
S_2	6(1)	4(2)	3(30)	5(1)	7(90)	120	$u_2=1$
S_3	5(30)	2(0)	6(3)	4(90)	8(1)	120	$u_3=1$
Demand	40	50	70	90	90		
v_j	$v_1=4$	$v_2=1$	$v_3=2$	$v_4=3$	$v_5=6$		

2. Find d_{ij} for all unoccupied cells (i,j), where $d_{ij} = c_{ij} - (u_i + v_j)$

$d_{14} = C_{14} - (u_1 + v_4) = 6 - (0+3) = 3$

$d_{15} = C_{15} - (u_1 + v_5) = 9 - (0+6) = 3$

$d_{21} = C_{21} - (u_2 + v_1) = 6 - (1+4) = 1$

$d_{22} = C_{22} - (u_2 + v_2) = 4 - (1+1) = 2$

$d_{24} = C_{24} - (u_2 + v_4) = 5 - (1+3) = 1$

$d_{32} = C_{32} - (u_3 + v_2) = 2 - (1+1) = 0$

$d_{33} = C_{33} - (u_3 + v_3) = 6 - (1+2) = 3$

$d_{35} = C_{35} - (u_3 + v_5) = 8 - (1+6) = 1$

Since all $d_{ij} \geq 0$

So, final optimal solution is arrived.

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	4(10)	1(50)	2(40)	6	9	100
S_2	6	4	3(30)	5	7(90)	120
S_3	5(30)	2	6	4(90)	8	120
Demand	40	50	70	90	90	

The minimum total transportation cost

$= 4 \times 10 + 1 \times 50 + 2 \times 40 + 3 \times 30 + 7 \times 90 + 5 \times 30 + 4 \times 90$

$= 1400/-$

Q. Obtain an optimum basic feasible solution to the transportation problem.

		Warehouse				
		W_1	W_2	W_3	W_4	Capacity
Factory	F_1	19	30	50	10	7
	F_2	70	30	40	60	9
	F_3	40	8	70	20	8
Requirement		5	8	7	14	

Sol:- No. of supply constraints = 3
No. of Demand constraints = 4.

Find IBFS using NWCM:

Problem table is :-

	D_1	D_2	D_3	D_4	Supply
S_1	19(5)	30(2)	50	10	7
S_2	70	30(6)	40(3)	60	9
S_3	40	8	70(4)	20(14)	18
Demand	5	8	7	14	

Solved.

The minimum transportation cost $= 19\times5 + 30\times2 + 30\times6 + 40\times3 + 70\times4 + 20\times14$
$= 1015/-$

Here, the no. of allocated cells $= 6 = m+n-1$
$\therefore$ The sol^n is non-degenerate.

Iteration - 1:-

Find u_i & v_j for all occupied cells (i,j) where $c_{ij} = u_i + v_j$.

1. Substituting $u_1 = 0$
2. $c_{11} = u_1 + v_1 \Rightarrow v_1 = 19$
3. $c_{12} = u_1 + v_2 \Rightarrow v_2 = 30$
4. $c_{22} = u_2 + v_2 \Rightarrow u_2 = 0$
5. $c_{23} = u_2 + v_3 \Rightarrow v_3 = 40$
6. $c_{33} = u_3 + v_3 \Rightarrow u_3 = 30$
7. $c_{34} = u_3 + v_4 \Rightarrow v_4 = -10$

2. Find d_{ij} for all unoccupied cells (i,j), where
$$d_{ij} = C_{ij} - (u_i + v_j)$$

i) $d_{13} = C_{13} - (u_1 + v_3) = 10$
ii) $d_{14} = C_{14} - (u_1 + v_4) = 20$
iii) $d_{21} = C_{21} - (u_2 + v_1) = 51$
iv) $d_{24} = C_{24} - (u_2 + v_4) = 70$
v) $d_{31} = C_{31} - (u_3 + v_1) = -9$
vi) $d_{32} = C_{32} - (u_3 + v_2) = -52$.

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19(5)	30(2)	50(10)	10(20)	7	$u_1 = 0$
S_2	70(51)	30(6)(−)	40(3)(+)	60(70)	9	$u_2 = 0$
S_3	40(−9)	8(−52)(+)	70(4)(−)	20(14)	18	$u_3 = 30$.
Demand	5	8	7	14		
v_j	$v_1 = 19$	$v_2 = 30$	$v_3 = 40$	$v_4 = -10$		

Now choose the minimum negative value from all d_{ij} (opportunity cost) $= d_{32} = [-52]$

Draw a closed path from S_3D_2.
Closed path $S_3D_2 \to S_3D_3 \to S_2D_3 \to S_2D_2$.

Minimum allocated value among all negative position (−) on closed path = 4.
Subtract 4 from all (−) & add to all (+)

	D_1	D_2	D_3	D_4	Supply.
S_1	19(5)	30(2)	50	10	7
S_2	70	30(2)	40(7)	60	9
S_3	40	8(4)	70	20(14)	18
Demand	5	8	7	14	

Iteration-2

1. Find u_i & v_j for all occupied cells (i,j), where $C_{ij} = u_i + v_j$.

1. Substituting, $v_2 = 0$
2. $C_{12} = u_1 + v_2 \Rightarrow u_1 = 30$
3. $C_{11} = u_1 + v_1 \Rightarrow v_1 = -11$
4. $C_{22} = u_2 + v_2 \Rightarrow u_2 = 30$
5. $C_{23} = u_2 + v_3 \Rightarrow v_3 = 10$
6. $C_{32} = u_3 + v_2 \Rightarrow u_3 = 8$
7. $C_{34} = u_3 + v_4 \Rightarrow v_4 = 12$

2. Find d_{ij} for all unoccupied cells (i,j), where $d_{ij} = C_{ij} - (u_i + v_j)$

1. $d_{13} = C_{13} - (u_1 + v_3) = 10$
2. $d_{14} = C_{14} - (u_1 + v_4) = -32$
3. $d_{21} = C_{21} - (u_2 + v_1) = 51$
4. $d_{24} = C_{24} - (u_2 + v_4) = 18$
5. $d_{31} = C_{31} - (u_3 + v_1) = 43$
6. $d_{33} = C_{33} - (u_3 + v_3) = 52$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 (5)	30 (2) (−)	50 (10)	10 (−32) (+)	7	$u_1 = 30$
S_2	70 (51)	30 (2)	40 (7)	60 (18)	9	$u_2 = 30$
S_3	40 (43)	8 (4) (+)	70 (52)	20 (14) (−)	18	$u_3 = 8$.
Demand	5	8	7	14		
v_j	$v_1 = -11$	$v_2 = 0$	$v_3 = 10$	$v_4 = 12$		

Now choose the minimum negative value from all d_{ij} (opportunity cost) = d_{14} = [-32]
& draw a closed path from S_1D_4.
Closed path is $S_1D_4 \rightarrow S_1D_2 \rightarrow S_3D_2 \rightarrow S_3D_4$.

Minimum allocated value among all negative position (-) on closed path = 2.

Subtract 2 from all (-) & add it to all (+)

	D_1	D_2	D_3	D_4	Supply
S_1	19 (5)	30	50	10 (2)	7
S_2	70	30 (2)	40 (7)	60	9
S_3	40	8 (6)	70	20 (12)	18
Demand	5	8	7	14	

Iteration-3.

1. Find u_i & v_j for all occupied cells (i,j), where $C_{ij} = u_i + v_j$

i) $u_1 = 0$

ii) $C_{11} = u_1 + v_1 \Rightarrow v_1 = 19$

iii) $C_{14} = u_1 + v_4 \Rightarrow v_4 = 10$

iv) $C_{34} = u_3 + v_4 \Rightarrow u_3 = 10$

v) $C_{32} = u_3 + v_2 \Rightarrow v_2 = -2$

vi) $C_{22} = u_2 + v_2 \Rightarrow u_2 = 32$

vii) $C_{23} = u_2 + v_3 \Rightarrow v_3 = 8$

2. Find d_{ij} for all unoccupied cells (i,j), where $d_{ij} = C_{ij} - (u_i + v_j)$

- $d_{12} = C_{12} - (u_1 + v_2) = 32$
- $d_{13} = C_{13} - (u_1 + v_3) = 42$
- $d_{21} = C_{21} - (u_2 + v_1) = 19$
- $d_{24} = C_{24} - (u_2 + v_4) = 18$
- $d_{31} = C_{31} - (u_3 + v_1) = 11$
- $d_{33} = C_{33} - (u_3 + v_3) = 52$

Since all $d_{ij} \geq 0$.

So final Optimal solⁿ is arrived.

	D_1	D_2	D_3	D_4	Supply
S_1	19(5)	30	50	10(2)	7
S_2	70	30(2)	40(7)	60	9
S_3	40	8(6)	70	20(12)	18
Demand	5	8	7	14	

The minimum total transportation cost

$= 19\times5 + 10\times2 + 30\times2 + 40\times7 + 8\times6 + 20\times12 = 743$ //.

DEGENERACY IN TRANSPORTATION PROBLEM:

In a transportation problem, the number of allocations should be equal to $(m+n-1)$, where m is no. of rows and n is no. of columns.

If these independent allocations will be less than $(m+n-1)$, then degeneracy will exist in the TBFS, & a solution is called degenerate. Allocate ε to the unoccupied cell which has least cost. ε is the smallest positive number (nearer to zero)

Ex:

	A	B	C	Supply
X	50	30	220	1
Y	90	45	170	3
Z	50	200	50	4
Demand	3	3	2	

Total supply = Total Demand

∴ The TP is balanced.

(P.T.O)

	A	B	C	Supply	RP_1	RP_2	RP_3	RP_4
X	50 (1)	30	220	~~1~~ 0	20	20	20	50
Y	90	45 (3)	170	~~3~~ 0	45	45	45	-
Z	50 (2)	200	50 (2)	4 ~~2~~ 0	0	150	-	-
Demand	~~3~~ ~~1~~ 0	~~3~~ 0	~~2~~ 0					
CP_1	0	15	$\boxed{120}$					
CP_2	0	15	-					
CP_3	40	15	-					
CP_4	50	-	-					

Transportation cost $= 50\times 1 + 45\times 3 + 50\times 2 + 50\times 2$

$= 50 + 135 + 100 + 100$

$= 385.$

No. of allocations $= 4$

$m+n-1 = 3+3-1 = 5$

The above solution is degenerate.

To make the soln non-degenerate we introduce artificial rim ε to any of the unoccupied cell, usually we choose minimum cost unoccupied cell.

	A	B	C	Supply	u_i
X	50 (1)	30 (ε)	220	1	$u_1 = 0$
Y	90	45 (3)	170	3	$u_2 = 15$
Z	50 (2)	200	50 (2)	4	$u_3 = 0$
Demand	3	3	2	8	
v_j	$v_1 =$	$v_2 =$	$v_3 =$	~~8~~	

For Optimality,

Find u_i & v_j for all occupied cells (i,j), where $c_{ij} = u_i + v_j$.

i) $u_1 = 0$

ii) $u_2 =$

iii) $u_3 = 0$

iv) $v_1 = 50$

v) $v_2 = 30$

vi) $v_3 = 50$

For unoccupied cells, $d_{ij} = c_{ij} - (u_i + v_j)$

$d_{13} = 170$
$d_{21} = 25$
$d_{23} = 105$
$d_{32} = 170$

$\Rightarrow$ We have already reached the stage where the transportation cost is minimum, which cannot be reduced further.

Since all $d_{ij} > 0$, the current sol^n is optimal.

$\therefore$ Min Transportation cost = 385 (as $\varepsilon \to 0$)

Q. Solve the transportation problem & obtain the optimal sol^n.

	D_1	D_2	D_3	Supply
S_1	2	2	3	10
S_2	4	1	2	15
S_3	1	3	1	40
Demand	20	15	30	65

Total Demand = Total Supply.
Apply NWCR for finding IBFS.

	D_1	D_2	D_3	Supply
S_1	$\cancel{2}$ 10	$\cancel{2}$	$\cancel{3}$	$\cancel{10}$ 0
S_2	$\cancel{4}$ 10	$\cancel{1}$ 5	$\cancel{2}$	$\cancel{15}$ $\cancel{5}$ 0
S_3	$\cancel{1}$	$\cancel{3}$ 10	$\cancel{1}$ 30	$\cancel{40}$ $\cancel{30}$ 0
Demand	$\cancel{20}$ $\cancel{10}$ 0	$\cancel{15}$ $\cancel{10}$ 0	$\cancel{30}$ 0	

The minimum transportation cost

$= 2\times10 + 4\times10 + 1\times5 + 3\times10 + 1\times30 = 20+40+5+30+30$
$= ₹125/-$

Since, no. of allocated cells $= m+n-1 = 5$
$\therefore$ Sol^n is non degenerate.

For Optimality,

① Find u_i & v_j for occupied cells (i,j), $c_{ij} = u_i + v_j$

	D_1	D_2	D_3	
S_1	2 (10)	2	3	$u_1 = 2$
S_2	4 (10)	1 (5)	2	$u_2 = 4$
S_3	1	3 (10)	1 (30)	$u_3 = 6$
	$v_1 = 0$	$v_2 = -3$	$v_3 = -5$	

② Find $d_{ij} = c_{ij} - (u_i + v_j)$ for unoccupied cell.

$d_{12} = c_{12} - (u_1 + v_2) = 3$

$d_{13} = 6$, $d_{23} = 3$, $d_{31} = -5$

$x = \min(10, 10) = 10$.

Iteration table :-

	D_1	D_2	D_3	
S_1	2 (10)	2 (ε)	3	$u_1 = 0$
S_2	4*	1 (15)	2	$u_2 = -1$
S_3	1 (10)	3	1 (30)	$u_3 = -1$
	$v_1 = 2$	$v_2 = 2$	$v_3 = 2$	

No. of allocations = 4

$m + n - 1 = 6 - 1 = 5$

∴ The sol^n is degenerate.

So, we add ε the smallest positive number to cell (1,2)

Find $c_{ij} = u_i + v_j$ for occupied cells,

Find $d_{ij} = c_{ij} - (u_i + v_j)$ for unoccupied cells.

$d_{13} = c_{13} - (u_1 + v_3) = 3 - (0+2) = 1$

$d_{21} = c_{21} - (u_2 + v_1) = 3$

$d_{23} = 1$

$d_{32} = 2$

Since all $d_{ij} \geq 0$, So we get the optimal solution.

Transportation cost $= 2\times10 + 2\times\varepsilon + 1\times15 + 1\times10 + 1\times30$

$= 20+15+10+30+2\varepsilon$

$= 75 + 2\varepsilon \quad (\varepsilon \to 0)$

$= ₹75$ /-

MAXIMIZATION CASE IN TRANSPORTATION PROBLEM.

If we have a transportation problem where the objective is to maximize the profit, first we have to convert the maximization problem in to a minimization problem by multiplying all the entries by (-1) or by subtracting all the entries from the highest entry in the given transportation table. The modified minimization problem can be solved in the usual manner.

Q: Solve the following transportation problem to maximize profit.

		D_1	D_2	D_3	D_4	Supply
	S_1	15	51	42	33	23
Source	S_2	80	42	26	81	44
	S_3	90	40	66	60	33
	Demand	23	31	16	30	

Sol:- Since the given problem is of maximization type, 1st convert this into a minimization problem by subtracting the cost element from the highest cost element $(c_{ij} = 90)$

in the given transportation problem.

	D_1	D_2	D_3	D_4	Supply.
S_1	75	39	48	57	23
S_2	10	48	64	9	44
S_3	0	50	24	30	33
Demand	23	31	16	30	

Find IBFS using VAM

	D_1	D_2	D_3	D_4	Supply	Row penalty					
S_1	~~75~~	39(17)	~~48~~(6)	~~57~~	~~23~~ ~~17~~ 0	9	9	9	9	9	39
S_2	~~10~~	~~48~~(14)	~~64~~	~~9~~(30)	~~44~~ ~~14~~ 0	1	[39]	16	16	-	-
S_3	~~0~~(23)	~~50~~	~~24~~(10)	~~30~~	~~33~~ ~~10~~ 0	[24]	6	[26]	-	-	-
Demand	~~23~~ 0	~~31~~ ~~14~~ 0	~~16~~ ~~6~~ 0	~~30~~ 0							
Column penalty	10	9	24	21							
	-	9	24	21							
	-	9	24	-							
	-	9	16	-							
	-	39	[48]	-							
	-	39	-	-							

Allocations in the original problem,

	D_1	D_2	D_3	D_4	Supply
S_1	15	51(17)	42(6)	33	23
S_2	80	42(14)	26	81(30)	44
S_3	90(23)	40	66(10)	60	33
Demand	23	31	16	30	

The maximum profit

$= 51\times17 + 42\times6 + 42\times14 + 81\times30 + 90\times23 + 66\times10 = 6867$

Here, the no. of allocated cells $= 6$ is equal to $m+n-1$

$= 3+4-1 = 6$ $\therefore$ This soln is non-degenerate

Optimality test using Modi method.

Allocation table is,

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	75	(+) 39(17)	(-) 48(6)	57	23	$u_1=0$
S_2	10 (+)	(-) 48(14)	64	9(30)	44	$u_2=9$
S_3	0(23) (-)	50	(+) 24(10)	30	33	$u_3=-24$
Demand	23	31	16	30		
v_j	$v_1=24$	$v_2=39$	$v_3=48$	$v_4=0$		

Iteration - 1 :-

1. Find u_i and v_j for all occupied cells (i,j), where $c_{ij}=u_i+v_j$

i. Substituting, $u_1=0$, we get.
ii. $C_{12}=u_1+v_2 \Rightarrow v_2=39$
iii. $C_{22}=u_2+v_2 \Rightarrow u_2=9$
iv. $C_{24}=u_2+v_4 \Rightarrow v_4=0$
v. $C_{13}=u_1+v_3 \Rightarrow v_3=48$
vi. $C_{33}=u_3+v_3 \Rightarrow u_3=-24$
vii. $C_{31}=u_3+v_1 \Rightarrow v_1=24$.

2. Find d_{ij} for all unoccupied cells (i,j), where $d_{ij}=C_{ij}-(u_i+v_j)$

$d_{11}=51$; $d_{14}=57$; $d_{21}=-23$; $d_{23}=7$; $d_{32}=35$; $d_{34}=54$

Now, choose the minimum negative value from all $d_{ij} \Rightarrow d_{21}=[-23]$

Draw the closed path from S_2D_1

Closed path is $S_2D_1 \rightarrow S_2D_2 \rightarrow S_1D_2 \rightarrow S_1D_3 \rightarrow S_3D_3 \rightarrow S_3D_1$

Minimum allocated value among all negative position (-) on closed path = 6.

Subtract 6 from all (-) & Add it to all (+)

	D_1	D_2	D_3	D_4	Supply
S_1	75	39 (23)	48	57	23
S_2	10(6)	48(8)	64	9 (30)	44
S_3	0(17)	50	24(16)	30.	33
Demand	23	31	16	30	

Iteration-2:-

1. Find u_i and v_j for all occupied cells (i,j), where $C_{ij} = u_i + v_j$

1) Substituting, $u_2 = 0$, we get
2) $C_{21} = u_2 + v_1 \Rightarrow v_1 = 10$
3) $C_{31} = u_3 + v_1 \Rightarrow u_3 = -10$
4) $C_{33} = u_3 + v_3 \Rightarrow v_3 = 34$
5) $C_{22} = u_2 + v_2 \Rightarrow v_2 = 48$
6) $C_{12} = u_1 + v_2 \Rightarrow u_1 = -9$
7) $C_{24} = u_2 + v_4 \Rightarrow v_4 = 9$

2. Find d_{ij} for all unoccupied cells (i,j), where $d_{ij} = C_{ij} - (u_i + v_j)$

$d_{11} = 74$; $d_{13} = 23$; $d_{14} = 57$; $d_{23} = 30$; $d_{32} = 12$; $d_{34} = 31$

Since all $d_{ij} \geq 0$.
So final optimal solution is arrived.

	D_1	D_2	D_3	D_4	Supply
S_1	75	39(23)	48	57	23
S_2	10(6)	48(8)	64	9(30)	44
S_3	0(17)	50	24(16)	30	33
Demand	23	31	16	30	

The Allocations in the Original problem.

	D_1	D_2	D_3	D_4	Supply
S_1	15	51(23)	42	33	23
S_2	80(6)	42(8)	26	81(30)	44
S_3	90(17)	40	66(16)	60	33
Demand	23	31	16	30	

The Maximum profit $= 51 \times 23 + 80 \times 6 + 42 \times 8 + 81 \times 30 + 90 \times 17 + 66 \times 16 = \underline{\underline{7005}}$

UNIT-2, C2- ASSIGNMENT PROBLEM

Suppose there are 'n' jobs to be performed and 'n' persons are available for doing these jobs. Assume that each person can do each job at a time but with varying degree of efficiencies.

Let C_{ij} be the cost if the i^{th} person is assigned to j^{th} job. The problem is to find an assignment so that the total cost of performing all the jobs is minimum problem of this kind, are known as assignment problem

→ Balanced Assignment problem.

If the no. of rows in the given cost matrix is equal to the no. of columns.

→ Unbalanced Assignment problem.

If the no. of rows in the given cost matrix is not equal to the no. of columns.

→ Mathematical formulation of an Assignment problem.

Consider an assignment problem of assigning 'n' jobs to 'n' machine. Let C_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and.

Let $x_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, & \text{if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine} \end{cases}$

$$\text{Minimize} = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_{ij}$$

Subject to constraints, $\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, 2, 3, \ldots, n$

$$\sum_{j=n}^{n} x_{ij} = 1, \; j = 1, 2, 3, \ldots n$$

and $x_{ij} = 0$ (or) 1

ASSIGNMENT ALGORITHM / HUNGARIAN METHOD :-

Step 1 :- Find out the each row minimum element and subtract it from that row.

Step 2 :- Find out the each column minimum element and subtract it from that column.

Step 3 :- Make assignment in the Opportunity cost table.

a) Identify rows with exactly one unmarked 0. Make an assignment to this single 0 by making a square ([0]) around it & cross of all other 0 in the same column.

b) Identify columns with exactly one unmarked 0. Make an assignment to this single 0 by making a square ([0]) around it & cross off all other 0 in the same rows.

c) If a row and/or column has two or more unmarked 0 and one cannot be chosen by inspection, then choose the cell arbitrarily.

d) Continue this process until all 0 in rows/columns are either assigned or cross of (⊗)

Step 4 :- If the number of assignments ([ ]) is equal to the order of the cost matrix an optimal solution is reached.
If the no. of assignments is than n (order), then go to next step.

Step 5:- Draw a set of horizontal & vertical lines to cover all the 0.

a) Tick(√) mark all the rows in which there is no assigned zero.

b) Examine tick (√) marked rows, if any 0 cell occurs in that row, then tick(√) mark that column.

c) Examine Tick(√) marked columns, if any assigned 0 exists in that columns, then tick (√) mark that row.

d). Repeat this process until no more rows or columns can be marked.

e). Draw a straight line for each unmarked rows & marked columns.

f). If the number of lines is equal to the no. of rows then the current solution is the Optimal, otherwise goto Step.-6.

Step-6:- Develop the new revised opportunity cost table

a) Select the minimum element, say k, from the cells not covered by any line,

b) Subtract k from each element not covered by a line.

c) Add k to each intersection element of two lines.

Step 7:- Repeat steps 3 to 6 until an optimal solution is arrived.

Q. Solve the assignment problem given below.

	A	B	C	D
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

Sol. No. of rows = No. of columns.
∴ It is balanced assignment problem.

Step 1:- Row reduced matrix.

	A	B	C	D
I	0	3	5	2
II	2	0	3	2
III	0	1	7	3
IV	3	2	3	0

Step 2:- Column reduced matrix

	A	B	C	D
I	0	3	2	2
II	2	0	0	2
III	0	1	4	3
IV	3	2	0	0

Step 3:- Make assignment in the Opportunity cost table

	A	B	C	D
I	[0]	3	2	2
II	2	[0]	$\cancel{0}$	2
III	$\cancel{0}$	1	4	3
IV	3	2	[0]	$\cancel{0}$

Step 4:- No. of assignments = 3.
no. of rows = 4
Which its not equal, so solution is not optimal.

Step(5) :- Draw a set of horizontal & vertical lines to cover all the 0.

	A	B	C	D	
I	$\boxed{0}$	3	2	2	✓(3)
II	2	$\boxed{0}$	$\cancel{0}$	2	
III	$\cancel{0}$	1	4	3	✓(1)
IV	3	2	$\boxed{0}$	$\cancel{0}$	
	✓(2)				

no. of lines ≠ no. of rows.

Step (6) :- Draw a revised table by selecting the smallest element, among the cells not covered by any line.

K=1, subtract 1 from every element in the cell not covered by a line.

Add K=1 to every element in the intersection cell of two lines.

	A	B	C	D
I	0	2	1	1
II	3	0	0	2
III	0	0	3	2
IV	4	2	0	0

Repeat steps 3 to 6.

	A	B	C	D
I	$\boxed{0}$	2	1	1
II	3	$\cancel{0}$	$\boxed{0}$	2
III	$\cancel{0}$	$\boxed{0}$	3	2
IV	4	2	$\cancel{0}$	$\boxed{0}$

No. of assignments = 4, number of rows = 4
which is equal, so solution is optimal.

Optimal solⁿ is,

	Cost
I → A	1
II → C	10
III → B	5
IV → D	5
Total	$\boxed{21}$

Q. Find solution of assignment problem using Hungarian method.

Work\Job	I	II	III
A	6	3	5
B	5	9	2
C	5	7	8

Sol.

No. of rows = No. of columns
∴ It is balanced assignment problem.

Step 1:- Row reduced matrix

	I	II	III
A	3	0	2
B	3	7	0
C	0	2	3

Step 2:- Column reduced matrix

	I	II	III
A	3	0	2
B	3	7	0
C	0	2	3

Step 3:- Make assignment.

	I	II	III
A	3	[0]	2
B	3	7	[0]
C	[0]	2	3

No. of assignments = no. of rows
So solution is optimal.

Cost

A → II = 3

B → III = 2

C → I = 5

Total $\boxed{10}$

Q. A departmental head as four subordinates, & four tasks to be performed. The subordinates differ in efficiency, & the tasks differ in their intrinsic difficulty, this estimate, of the time each man would take to perform each test, is given in this matrix below. How should the tasks be allocated, one to a man, so as to minimize the total man-hours.

Men.

Tasks	I	II	III	IV
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

Sol:-

18	26	17	11
13	28	14	26
38	19	18	15
19	26	24	10

Step1:- Row reduced matrix.

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

Step2:- Column Reduced matrix

7	11	5	0
0	11	0	13
23	0	2	0
9	12	13	0

Step 3:- Make assignment in the Opportunity class table.

7	11	5	[0]
[0]	11	$\cancel{0}$	13
23	[0]	2	$\cancel{0}$
9	12	13	$\cancel{0}$

No. of assignments $\neq$ No. of rows.

Step 4:- • Mark row(4) since there is no assignment.
• Mark column(4) since row (4) has 0 in this column.
• Mark row which has assigned [0] in marked row.

7	11	5	[0]	✓
~~[0]~~	~~11~~	~~$\cancel{0}$~~	~~13~~	
~~23~~	~~[0]~~	~~2~~	$\cancel{0}$	
9	12	13	$\cancel{0}$	✓
			✓	

Step5:- Develop the new revised opportunity cost table.
K=5, Subtract k from every element in the cell not covered by a line.
Add k to every element in the intersection cell of two lines.

	I	II	III	IV
A	2	6	[0]	$\cancel{0}$
B	[0]	11	$\cancel{0}$	18
C	23	[0]	2	5
D	4	7	8	[0]

A → III, B → I, C → II, D → IV
$\Rightarrow$ 17 + 13 + 19 + 10 = 59 man-hours.

Q. Work / Job 1 2 3 4

Work / Job	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

Given;

10	12	19	11
5	10	7	8
12	14	13	11
8	15	11	9.

Cost

A → 2 → 12

B → 3 → 7

C → 4 → 11

D → 1 → 8

Total = 38.

Q. Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows,

		Job 1	2	3	4	5
Person	A	8	4	2	6	1
	B	0	9	5	5	4
	C	3	8	9	2	6
	D	4	3	1	0	3
	E	9	5	8	9	5

Sol: Since no. of rows = no. of columns
The given assignment problem is balanced.

Step 1:- Row reduced matrix.

$$\begin{matrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{matrix}$$

Step 2:- Column reduced matrix

$$\begin{matrix} 7 & 3 & 0 & 5 & 0 \\ 0 & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 4 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 4 & 0 \end{matrix}$$

Step 3:- Make assignment in oppurtunity cost matrix.

$$\begin{matrix} 7 & 3 & 0 & 5 & [0] \\ [0] & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & [0] & 4 \\ 4 & 3 & [0] & 0 & 3 \\ 4 & [0] & 2 & 4 & 0 \end{matrix}$$

no. of assignments = no. of rows.
∴ The solu is optimal.

Person		Job	Cost
A	→	5	1
B	→	1	0
C	→	4	2
D	→	3	1
E	→	2	5

∴ Total cost = ₹9

Q. The processing time in house for the jobs when allocated to the different machines are indicated below. Assign the machines for the jobs so that the total processing time is minimum.

Machines.

Jobs		M_1	M_2	M_3	M_4	M_5
	J_1	9	22	58	11	19
Jobs	J_2	43	78	72	50	63
	J_3	41	28	91	37	45
	J_4	74	42	27	49	39
	J_5	36	11	57	22	25

Sol:- Since the no. of rows = no. of columns, The given assignment problem is balanced.

Step 1:- Row reduced matrix

0	13	49	2	10	(9)
0	35	29	7	20	(43)
13	0	63	9	17	(28)
45	15	0	22	12	(27)
25	0	46	11	14	(11)

Step 2: Column reduced matrix.

0	13	49	0	0
0	35	29	5	10
13	0	63	7	7
45	15	0	20	2
25	0	46	9	4

Step 3: Make assignment in Opportunity cost table.

0	13	49	[0]	0
[0]	35	27	5	10
13	[0]	63	7	7
45	15	[0]	20	2
25	0	46	9	4

no. of assignments ≠ no. of rows.

minimum unmarked element = 2

- Subtract '2' from all other unmarked elements
- Add '2' to elements which are at intersections

Step 4:-

2	15	51	[0]	0
[0]	35	29	3	8
13	[0]	63	5	5
45	15	[0]	18	0
25	0	46	7	2

Step 5. ← Now, make assignments

no. of assignments ≠ no. of rows.

minimum unmarked element is 2.

- Subtract 2 from unmarked elements
- add 2 to intersected cells.

Step 6:-

4	17	51	[0]	0
[0]	35	27	1	6
13	[0]	61	3	3
45	17	[0]	18	0
25	0	44	5	[0]

no. of assignments = no. of rows.

∴ Current sol^n is Optimal.

Jobs	Machines	Cost/Time
J_1	M_4	11
J_2	M_1	43
J_3	M_2	28
J_4	M_3	27
J_5	M_5	25

Total cost = 134 hours

Unbalanced Assignment Problem.

If the number of rows is not equal to the number of columns in the cost matrix of the given assignment Problem, then the given assignment problem is said to be unbalanced.

First convert the unbalanced assignment problem into a balanced one by adding dummy rows or dummy columns with zero cost elements in the cost matrix depending upon whether $m<n$ or $m>n$ and then solve by the usual method.

ex 1:- A company has four machines to do 3 jobs each job can be assigned to one & only one machine. The cost of each job on each machine is given in the following table.

		Machines			
		1	2	3	4
Jobs	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

What are job assignments which will minimize the cost?

Sol:- The cost matrix of the given assignment problem is,

18	24	28	32
8	13	17	19
10	15	19	22

Since the number of rows is less than the no. of columns in the cost matrix the given assignment problem is unbalanced.

To make it a balanced one, add a dummy job row with zero cost elements. The balanced cost matrix is given by,

18	24	28	32
8	13	17	19
10	15	19	22
0	0	0	0

Step 1:- Row reduced matrix.

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

Step 2:- Column reduced matrix

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

Step 3 :- Make assignments in cost matrix.

$$\begin{matrix} \boxed{0} & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & \boxed{0} & 0 & 0 \end{matrix}$$

No. of assignments $\neq$ no. of rows

$\therefore$ The current soln is not optimal.

Choose the minimum in unmarked cells and subtract this minimum element from all unmarked cells.

Add this minimum to the element which is at intersection. [$\because$ minimum element = 5]

$$\begin{matrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{matrix}$$

Again make the assignments.

$$\begin{matrix} \boxed{0} & 1 & 5 & 9 \\ 0 & \boxed{0} & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & \boxed{0} & 0 \end{matrix}$$

no. of assignments $\neq$ no. of rows.

$\therefore$ The current solution is not optimal.

Choose minimum from unmarked cells

minimum element = 4.

$$\begin{matrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 9 & 4 & 0 & 0 \end{matrix}$$

Make assignments in the cost matrix

$$\begin{matrix} [0] & 1 & 1 & 5 \\ 0 & [0] & 0 & 2 \\ 0 & 0 & [0] & 3 \\ 9 & 4 & 0 & [0] \end{matrix}$$

→ Diagonal assignments of zero's.

no. of assignments = no. of rows

∴ The current sol^n is optimal.

Job		Machines	Cost.
A	→	1	18
B	→	2	13
C	→	3	19
D	→	4	0

Optimum Assignment cost = 50/- units of cost.

Note:- For this problem, the alternative optimum schedule is A → 1, B → 3, C → 2, D → 4, with the same Optimum cost = Rs (18 + 17 + 15 + 0) = 50/- units of cost.

Assign four trucks 1, 2, 3 and 4 to vacant spaces A, B, C, D, E and F so that the distance travelled is minimized. The matrix show the distance

		Trucks			
		1	2	3	4
Spaces	A	4	7	3	7
	B	8	2	5	5
	C	4	9	6	9
	D	7	5	4	8
	E	6	3	5	4
	F	6	8	7	3

Since the no. of rows ≠ no. of columns, the assignment problem is unbalanced.

To make it balanced, we introduce two dummy trucks (columns) with zero costs, we get

4	7	3	7	0	0
8	2	5	5	0	0
4	9	6	9	0	0
7	5	4	8	0	0
6	3	5	4	0	0
6	8	7	3	0	0

Step 1:- Row reduced matrix.

4	7	3	7	0	0
8	2	5	5	0	0
4	9	6	9	0	0
7	5	4	8	0	0
6	3	5	4	0	0
6	8	7	3	0	0

Step 2: Column Reduced matrix.

0	5	0	4	0	0
4	0	2	2	0	0
0	7	3	6	0	0
3	3	1	5	0	0
2	1	2	1	0	0
2	6	4	0	0	0

Step 3: Make Assignments.

0	5	[0]	4	0	0
4	[0]	2	2	0	0
[0]	7	3	6	0	0
3	3	1	5	[0]	0
2	1	2	1	0	[0]
2	6	4	[0]	0	0

Since each row & each column contains exactly one assignment, the current assignment is optimal.

∴ The Optimum assignment schedule is given by,

$A \to 3$, $B \to 2$, $C \to 1$, $D \to 5$, $E \to 6$, $F \to 4$

Optimum distance $= 3+2+4+0+0+3$

$= 12$/- units of distance.

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Maximization Case in Assignment Problems.

In a maximization assignment problem, the goal is to assign tasks to agents or Jobs to machines in a way that maximizes the overall profit or benefit.

To solve a maximization problem using Hungarian method, we often convert it into a minimization problem. This is done by subtracting each value in the cost matrix from the highest value in that matrix.

Q1. A Company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman & the nature of districts the company estimates that the profit per day in rupees for each salesman in each district is as below.

		Districts			
		1	2	3	4
	A	16	10	14	11
Salesmen	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	13

Find the assignment of salesmen to various districts which yield maximum profit.

Sol: The cost matrix of the given transportation problem is,

16	10	14	11
14	11	15	15
15	15	13	12
13	12	14	13.

This is a maximization problem, convert this into equivalent minimization problem by subtract each element of the cost matrix by highest element '16'

0	6	2	5
2	5	1	1
1	1	3	4
3	4	2	3

Step 1:- Row reduced matrix.

0	6	2	5
1	4	0	0
0	0	2	3
1	2	0	1

Step 2:- Column Reduced matrix.

0	6	2	5
1	4	0	0
0	0	2	3
1	2	0	1

Step 3:- Make assignments.

[0]	6	2	5
1	4	0	[0]
0	[0]	2	3
1	2	[0]	1

no. of assignments = no. of rows

∴ Current assignment is Optimum.

A → 1   16
B → 4   15
C → 2   15
D → 3   14

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Restricted Assignment Problem :- It is a variation of the classic assignment problem in which certain tasks cannot be assigned to specific agents due to constraints or limitations. The objective is still to minimize the total cost, while respecting these restrictions.

Q Five workers are available to work with the machines and the respective costs (in rupees) associated each worker machine assignment is given below.

	M_1	M_2	M_3	M_4
W_1	8	14	10	12
W_2	-	16	14	8
W_3	6	-	10	6
W_4	12	12	8	4

Worker W_2 cannot be assigned with machine M_1
Worker W_3 cannot be assigned with machine M_2

8	14	10	12
∞	16	14	8
6	∞	10	6
12	12	8	4

i) Row reduced matrix.

0	6	2	4
∞	8	6	0
0	∞	4	0
8	8	4	0

ii) Column reduced matrix.

0	0	0	4
∞	2	4	0
0	∞	2	0
8	2	2	0

iii) Make assignments.

$$\begin{matrix} 0 & \boxed{0} & 0 & 4 \\ \infty & 2 & 4 & \boxed{0} \\ \boxed{0} & \infty & 2 & 0 \\ 8 & 2 & 2 & 0 \end{matrix}$$

no. of Assignments ≠ no. of rows.

iv) Subtract minimum from all unmarked elements. Add minimum at intersections

$$\begin{matrix} 2 & 0 & 0 & 6 \\ \infty & 0 & 2 & 0 \\ 0 & \infty & 0 & 0 \\ 8 & 0 & 0 & 0 \end{matrix}$$

v) Again make assignments.

$$\begin{matrix} 2 & \cancel{0} & \boxed{0} & 6 \\ \infty & \boxed{0} & 2 & \cancel{0} \\ \boxed{0} & \infty & 0 & 0 \\ 8 & \cancel{0} & \cancel{0} & \boxed{0} \end{matrix}$$

$W_1 \rightarrow M_3 = 10$
$W_2 \rightarrow M_2 = 16$
$W_3 \rightarrow M_1 = 6$
$W_4 \rightarrow M_4 = 4$
Rs 36

No. of Assignments = no. of rows

∴ Current assignment is Optimal