Mathematical Logic.

Introduction

Logic is the study and analysis of the nature of the valid argument.

Propositions :-

A declarative sentence which is true or false but not both is called a proposition or a statement.

Examples of propositions :-

- 9 is less than 6
- $4+9<6$

Examples which is not a proposition :-

- "A is less than 2"
- Oh! What a beautiful flower it is?

We usually denote the statements by small letter p, q, r, etc.

Truth value :- The truthfulness or the falsity of a proposition is called the truth value of the proposition.

→ If a proposition is true then its truth value is denoted by T and if it is false then we say its truth value is F.

Exercise

1. Decide which of the following are propositions.

a) 475 is an even integer - Yes

b) Bangalore is the capital of Tamil Nadu - Yes

c) He has constructed a beautiful house - No

d) Sum of an even and odd integer is an even number - Yes

e) Bring me a glass of water - No

Logical connectives

Connectives are logical symbols that express the relationship between propositions.

Some common connectives are:-

- Conjunction -
  Symbolized by $\wedge$, this means 'and'.
- Disjunction -
  Symbolized by $\vee$, this means 'or'.
- Negation -
  Symbolized by $\sim$, this means 'not'.
- Conditional -
  Symbolized by $\rightarrow$, this means 'if... then'

- Biconditional :-
  Symbolized by $\leftrightarrow$, this means 'if and only

Compound Propositions

A proposition containing one or more connectives is called a compound proposition. The statement used to construct a compound proposition are called components

Exercise.

1. Symbolise the following propositions.

(a) $3x=9$ and $x<7$

Let $p: 3x=9$ and $q: x<7$.

$\therefore p \wedge q$

(b) $y+4 \neq 4$ and $y \nless 3$

Let $p: y+4=4$ and $q: y<3$

$\therefore \sim p \wedge q$

(c) If water is dry then snow is hot.

Let p: water is dry and q: Snow is hot

$\therefore$ Proposition is $p \rightarrow q$.

(d) If two numbers are not equal then their squares are not equal.

Sol: Let p: two numbers are equal
q: Squares of the numbers are equal

$\therefore$ Proposition is $\sim p \rightarrow \sim q$.

(e) He is neither intelligent nor industrious

Let p: He is intelligent
q: He is industrious.

$\sim(p \vee q)$ or $\sim p \wedge \sim q$.

Negation :- A proposition obtained by inserting the word 'not' at an appropriate place in a given proposition is called the negation of the given proposition. Denoted by $\sim p$ or $\neg p$ (read" not p')

ex:- Let p: Bangalore is a city.
$\sim p$: Bangalore is not a city.

Truth Table :-

p	$\sim p$
T	F
F	T

Conjunction

If two statements are combined by the word "and" to form a compound proposition is called conjunction.

Symbolically, if p and q are any two propositions Then conjuction of p and q is denoted by $p \wedge q$

→ Conjunction $p \wedge q$ is true only when both p and q are true, Otherwise false.

Truth table :-

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:-

Two statements are combined by the word 'or' to form a compound proposition is called Disjunction.

Symbolically, if p and q are any two propositions Then disjunction of p and q is denoted by $p \vee q$.

→ Disjunction $p \vee q$ is false if both p and q are false and in Other cases it is true.

Truth table :-

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Date :
Page No :

Conditional :- (Implication)

If p and q be two propositions, the compound proposition "if p then q" is known as the conditional of p and q and it is denoted by $p \to q$.

[p is called hypothesis (antecedent), q is called a conclusion]

→ The conditional $p \to q$ is false whenever p is true and q is false

p	q	$p \to q$	converse $q \to p$	[contrapositive $\sim q \to \sim p$]
T	T	T	T	
T	F	F	T	
F	T	T	F	
F	F	T	T	

Biconditional :- (Double implication)

If p and q be two propositions, the compound proposition "p if and only if q" is called a Biconditional proposition, and is denoted by $p \leftrightarrow q$

* $p \leftrightarrow q = (p \to q) \wedge (q \to p)$

→ The biconditional $p \leftrightarrow q$ is true only when p and q are both true or both false.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Exercise

1. If the truth values of the propositions p, q, r are respectively F, F, T find the truth value of each of the following compound propositions.

(i) $(p \vee q) \wedge r$

p	q	r	$(p \vee q)$	$(p \vee q) \wedge r$
F	F	T	F	F

(ii) $(p \rightarrow q) \wedge r$

p	q	r	$(p \rightarrow q)$	$(p \rightarrow q) \wedge r$
F	F	T	T	T

(iii) $(p \rightarrow q) \vee r$

p	q	r	$(p \rightarrow q)$	$(p \rightarrow q) \vee r$
F	F	T	T	T

(iv) $(p \vee r) \rightarrow q$

p	q	r	$(p \vee r)$	$(p \vee r) \rightarrow q$
F	F	T	T	F

(v) $(p \vee r) \rightarrow \sim q$

p	q	r	$(p \vee r)$	$\sim q$	$(p \vee r) \rightarrow \sim q$
F	F	T	T	T	T

2 (i) If $p \rightarrow (q \vee r)$ is false, find the truth values of p, q, r

$p \rightarrow (q \vee r)$	p	$(q \vee r)$	q	r
F	T	F	F	F

$\therefore$ T, F, F.

(ii) If $(p \wedge q) \to r$ is false, find the truth value of p, q, r

$(p \wedge q) \to r$	$(p \wedge q)$	r	p	q
F	T	F	T	T

$\therefore$ p, q, r are T, T, F respectively.

(iii) If $(p \wedge q) \vee [(\sim(p \wedge q)]$ is true, find the truth values of p and q.

$(p \wedge q) \vee [(\sim(p \wedge q)]$		$(p \wedge q)$	$\sim(p \wedge q)$	p	q
T		T	F	T	T
	or	F	T	or T	F
	or	T	T	or F	F

3. Write the truth table for each of the following

i) $\sim(p \wedge q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

ii) $(p \wedge q) \vee r$

(P.T.O)

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

(iii) $(p \vee q) \vee \sim p$

p	q	$\sim p$	$(p \vee q)$	$(p \vee q) \vee \sim p$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

(iv) $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$\sim p \wedge \sim q$	$(\sim p \vee q$
T	T	F	F	T	F	F
T	F	F	T	F	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

<u>Converse</u>, <u>Inverse</u> and <u>Contrapositive</u>.

1. Converse :- If the given conditional is $p \to q$ then
The converse of this conditional is $q \to p$

2. Inverse : If the given conditional is $p \to q$ then the inverse of this conditional is $\sim p \to \sim q$

3. Contrapositive : If the given conditional is $p \to q$ then the contrapositive of this conditional is $\sim q \to \sim p$.

Logical Equivalence

Two statements are said to be logically equivalent if they have the same truthvalues for all logical possibilities.

Whenever two statements S_1 and S_2 are logically equivalent we write $S_1 \equiv S_2$ and read as S_1 is logically equivalent to S_2.

ex:-

(i) $p \wedge q \equiv q \wedge p$

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$\therefore (p \wedge q) \equiv (q \wedge p)$

(ii) $(p \wedge q) \equiv \sim(p \to \sim q)$

p	q	$\sim q$	$p \wedge q$	$p \to \sim q$	$\sim(p \to \sim q)$
T	T	F	T	F	T
T	F	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F

$\therefore p \wedge q \equiv \sim(p \to \sim q)$

(iii) $\sim(\sim p) \equiv p$

① p	② $\sim p$	③ $\sim(\sim p)$
T	F	T
F	T	F

1^{st} & 3^{rd} column are identical
$\therefore \sim(\sim p) \equiv p$.

(iv) $\sim(p \wedge q) \equiv \sim p \vee \sim q$ Demorgan's law

① p	② q	③ $p \wedge q$	④ $\sim(p \wedge q)$	⑤ $\sim p$	⑥ $\sim q$	⑦ $\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

4^{th} & 7^{th} column are identical
$\therefore \sim(p \wedge q) = \sim p \vee \sim q$

Tautology and Contradiction.

Tautology :- A compound proposition is said to be a Tautology, if it is always true for all possible combinations of the truth values of its components.

Contradiction :- A compound proposition which is always false for all possible combinations of the truth values of its components, is called a contradiction.

ex:- Show that the proposition, $(p \to q) \leftrightarrow \sim p \vee q$ is a tautology.

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow \sim p \vee q$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

The last column indicates that the given proposition is always true. Hence the given statement is tautology.

ex:- Show that the proposition p and q, $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

The last column indicates that the given proposition is always false. Hence the given statement is contradiction.

Exercise.

1. Determine whether each of the following is a tautology, a contradiction or neither.

(a) $(p \vee q) \wedge (\sim p \wedge \sim q)$

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

$\therefore$ Contradiction

(b) $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

$\therefore$ Tautology

(c) $(c \wedge p) \leftrightarrow c$, where c is a contradiction.

c	p	$c \wedge p$	$(c \wedge p) \leftrightarrow c$
F	T	F	T
F	F	F	T

$\therefore$ Tautology

(d) $(t \wedge p) \leftrightarrow p$, where t is a tautology.

t	p	$t \wedge p$	$(t \wedge p) \leftrightarrow p$
T	T	T	T
T	F	F	T

$\therefore$ Tautology

Laws of logic.

Name	Equivalence
1. Identity laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
2. Domination laws	$p \vee T \equiv T$ $p \wedge F \equiv F$
3. Law of Double negation	$\sim(\sim p) \equiv p$
4. Idempotent laws	$p \wedge p \equiv p$ $p \vee p \equiv p$
5. Commutative laws	$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$
6. Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
7. Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
8. De Morgan's laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$ $\sim(p \vee q) \equiv \sim p \wedge \sim q$
9. Absorption laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
10. Negation laws	$p \vee \sim p \equiv T$ $p \wedge \sim p \equiv F$
11. Equivalent law	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Counting.

Combinatorics deals with counting and enumeration of specified objects, patterns or designs. Combinatorics includes the study of permutations, combinations, and partitions. is concerned with determining the number of logical possibilities of some event without necessarily identifying every case.

Basics of Counting.

The Rule of Sum and Rule of Product are used to decompose difficult counting problems into simple problems.

- The Rule of Sum - If a sequence of tasks $T_1, T_2, \dots T_m$ can be done in $w_1, w_2, \dots, w_m$ ways respectively (the condition is that no tasks can be performed simultaneously), then the number of ways to do one of these tasks is $w_1 + w_2 + \dots + w_m$. If we consider two tasks A and B which are disjoint (i.e. $A \cap B = \phi$), then mathematically, $|A \cup B| = |A| + |B|$

(or = +)

$$|A + B| = |A| + |B|$$

- The Rule of Product - If a sequence of tasks $T_1, T_2, \dots, T_m$ can be done in $w_1, w_2, \dots, w_m$ ways respectively and every tasks arrives after the occurence of the previous task, then there are $w_1 \times w_2 \times w_3 \times \dots \times w_m$ ways to perform the tasks. Mathematically, If a task B arrives after a task A, then $|A \times B| = |A| \times |B|$

(and = ×)

ex1: In Bangalore city, 8 newspaper and 4 magazines are printed. Rajesh wants to subscribe 1 newspaper or 1 magazine. How many choices does he have.

Ans: $8+4 = 12$ choices

In 12 ways Rajesh can subscribe a newspaper or a magazine.

ex2: In how many ways can we draw a club or a diamond from a pack of cards?

Ans: [A deck of cards consists of 52 cards divided into 4 suits: hearts, clubs, spades and diamonds]

There are 13 club and 13 diamonds in a pack of cards

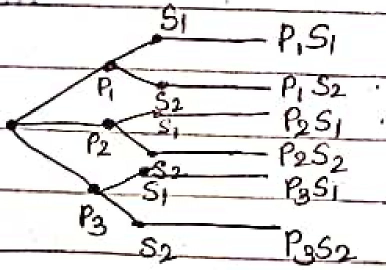
The number of ways a club or diamond may be drawn $13+13 = 26$.

ex3: Mohan has 3 pants and 2 shirts. In how many different ways can Mohan dress up with?

Ans: Since Mohan has 3 pants and 2 shirts

Total number of ways $= 3 \times 2 = 6$

In 6 ways he can dressup

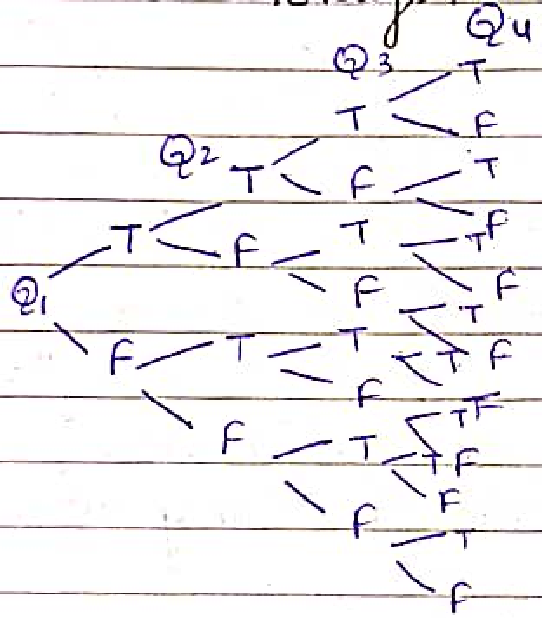


4. In how many different ways one can answer all the questions of a true - false testing consisting of 4 question.

Ans: There are two ways of answering each of the 4 questions.

$\therefore$ By Product rule, the number of ways in which all the 4 questions can be answered

$= 2^4 = 16$ ways.



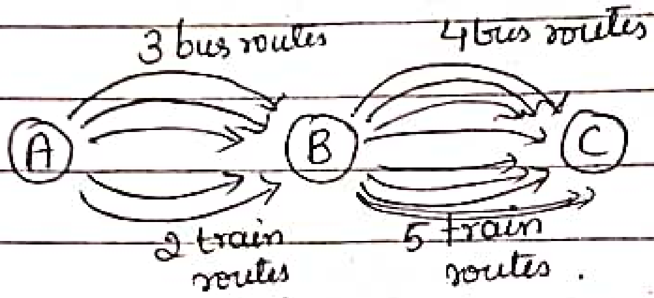
5. How many different two wheeler licence plates can be made if each plate contains a sequence of two uppercase English letters followed by four digits (Alphabets and digit repeating is permissible)?

Sol:- There are 26 choices for each of the two uppercase English letters and ten choices for each of the four digits.

Hence, by the product rule there are a total of $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 67,60,000$ possible licence plates.

6. A boy lives at A and wants to go to school at C. From his home A he has to first reach B and then B to C. He may go A to B by either 3 bus routes or 2 train routes. From there, he can either choose 4 bus routes or 5 train routes to reach C. How many ways are there to go from A to C?

Sol:-



$3+2=5$ ways $\quad 4+5=9$ ways.

Hence from A to C he can go in $5 \times 9 = 45$ ways.

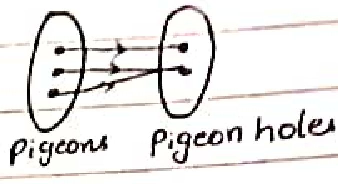
7. A Wardrobe consists of 6 shirts, 5 pairs of pants, and 19 bow ties. How many different outfits can you make?

Sol: There are $6 \times 5 \times 19 = 570$ outfits.

PIGEONHOLE PRINCIPLE :-

The Pigeonhole principle can be stated as,
" If 'n' items are distributed among 'm' containers and $n > m$, then atleast one container must contain more than one item".

The Generalized Pigeonhole principle :



If 'N' objects are placed into 'k' boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Note :- $\lceil\ \rceil$ is the ceiling function. i.e., always we will take next integer.

1] ex:- Find minimum number of students in a class, so that two students were born in same month.

Sol:- Let 'n' be the no. of students (Pigeons)
k be the no. of months in a year (pigeon holes)
∴ minimum no. of students in a class = 13.

$n > k$, $\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{13}{12} \right\rceil = \lceil 1.08 \rceil = 2$

2] If 6 colour are used to paint 37 home. Show that atleast 7 home of them will be of same colour.

Sol:- $n = 37$
$k = 6$

∴ No. of homes which are in same colour $= \left\lceil \frac{37}{6} \right\rceil$
$= \lceil 6.166 \rceil$
$= 7$

Permutation :-

Permutation is an arrangement of objects in which the order does matter. The number of permutations of 'n' objects chosen 'r' at a time is denoted by ^nP_r, where $0<r\leq n$

→ The number of permutations of 'n' objects taken 'r' at a time is given by $^nP_r = \frac{n!}{(n-r)!}$

→ If $r=n$, then $^nP_n = n!$

→ If $r=0$, then $^nP_0 = 1$.

→ The no. of permutations of 'n' different objects taken 'r' at a time whose repetition is allowed is n^r.

→ The number of permutations of n objects, where p objects are of the same kind and rest are all different $= \frac{n!}{p!}$

→ The number of permutations of n objects, where P_1 objects are of the first kind, P_2 objects are of the second kind, ... P_K are of the Kth kind and the rest if any, are of different kind is

$$\frac{n!}{P_1!P_2!\cdots P_n!}$$

→ Clue words :- arrangement, Schedule, order.

Factorial notation :-

The notation $n!$ represents product of first n natural numbers.

* $n! = n(n-1)(n-2) \times \dots \times 3 \times 2 \times 1$
* $n! = n(n-1)!$ * $0! = 1! = 1$

Worked Problems.

1. Evaluate $7! - 5!$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$\therefore 7! - 5! = 5040 - 120$

$= 4920$

2. Find the value of

i) ${}^{12}P_4$ ii) ${}^{7}P_3$

$${}^{12}P_4 = \frac{(12)!}{(12-4)!}$$

$$= \frac{(12)!}{8!}$$

$$= \frac{12 \times 11 \times 10 \times 9 \times 8!}{8!}$$

$$= 11880$$

$${}^{7}P_3 = \frac{7!}{4!}$$

$$= 7 \times 6 \times 5$$

$$= 210$$

Date :
Page No :

3. Find the value of n, if ${}^{n}P_2 = 12$

$${}^{n}P_2 = 12$$

$$\Rightarrow \frac{n!}{(n-2)!} = 12$$

$$\Rightarrow \frac{n(n-1)(n-2)!}{(n-2)!} = 12$$

$$\Rightarrow n^2 - n = 12$$

$$\Rightarrow n^2 - n - 12 = 0$$

$$\Rightarrow (n-4)(n+3) = 0$$

$$\Rightarrow \boxed{n=4} \text{ or } \boxed{n=-3}$$

Since n cannot be negative

$$\boxed{n=4}$$

4. Find n, if ${}^{n-1}P_3 : {}^{n+1}P_3 = 5:12$

$$\frac{{}^{n-1}P_3}{{}^{n+1}P_3} = \frac{5}{12}$$

$$\Rightarrow \frac{(n-1)!}{(n-1-3)!} \times \frac{(n+1-3)!}{(n+1)!} = \frac{5}{12}$$

$$\Rightarrow \frac{(n-1)!}{(n-4)!} \times \frac{(n-2)!}{(n+1)!} = \frac{5}{12}$$

$$\Rightarrow \frac{\cancel{(n-1)!}}{\cancel{(n-4)!}} \times \frac{(n-2)(n-3)\cancel{(n-4)!}}{(n+1)\,n\,\cancel{(n-1)!}} = \frac{5}{12}$$

$$\Rightarrow \frac{n^2 - 3n - 2n + 6}{n^2 + n} = \frac{5}{12}$$

$$\Rightarrow \frac{n^2 - 5n + 6}{n^2 + n} = \frac{5}{12}$$

$$\Rightarrow 12n^2 - 60n + 72 = 5n^2 + 5n$$

$$\Rightarrow 12n^2 - 5n^2 - 60n - 5n + 72 = 0$$

$$\Rightarrow 7n^2 - 65n + 72 = 0$$

Date :
Page No :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 7$, $b = -65$, $c = 72$

$$x = \frac{65 \pm \sqrt{(65)^2 - 4(7)(72)}}{2(7)}$$

$$x = \frac{65 \pm \sqrt{4225 - 2016}}{14}$$

$$x = \frac{65 \pm \sqrt{2209}}{14}$$

$$x = \frac{65 \pm 47}{14}$$

$$x = \frac{65 + 47}{14} \quad \text{or} \quad x = \frac{65 - 47}{14}$$

$$x = \frac{112}{14} \quad \text{or} \quad x = \frac{18}{14}$$

$$x = 8 \quad \text{or} \quad x = \frac{9}{7}$$

5. How many 3 digit numbers can be formed by using the digits 1 to 9 (i) if no digit is repeated? (ii) if repetition is allowed?

Sol:- (i) There are 9 numbers 1, 2, 3, ..., 9

$\therefore n = 9$

3 digits number should be formed

$r = 3$

$$\therefore {}^nP_r = {}^9P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!}$$

$$= \frac{9 \times 8 \times 7 \times 6!}{6!}$$

$$= 9 \times 8 \times 7 = 504.$$

(OR) $\underset{9\text{ ways}}{\square} \times \underset{8\text{ ways}}{\square} \times \underset{7\text{ ways}}{\square} = 504$

ii) When repetition is allowed number of permutations is $n^r = 9^3 = 729$

(OR) $\boxed{}\ \boxed{}\ \boxed{}$

$9 \times 9 \times 9 = 729$.

6. How many 3-digit even numbers can be made using the digits 1,2,3,4,6,7 if no digit is repeated.

Sol: $\boxed{}\ \boxed{}\ \boxed{}$

$\underset{\text{ways}}{5} \times \underset{\text{ways}}{4} \times \underset{\text{3ways}}{\{2,4,6\}} = 60$.

(OR) Since 3 digit no. is even, unit place should be even, To fill unit place we have 3 digits from which one has to be chosen

$\therefore {}^3P_1 = 3$

Hundreds & Tens place can be fixed using 5-digits.

$5P_2 = 20$.

$\therefore$ 3-digit even numbers that can be made is $20 \times 3 = 60$ ways.

7. How many numbers greater than a million can be formed by using the digits, 4,6,0,6,7,4,6

Sol: A million is a seven digit number, and the digits to be used are 4,6,0,6,7,4,6. We see 4 occurs twice, 6 occurs thrice and 0,7 occur once.

Date :
Page No :

$\therefore$ Total number of arrangements $= \frac{7!}{2!\,3!} = \frac{5040}{2\times6} = 42$

The numbers formed of these seven digits beginning with '0' are not greater than one million.

Number beginning with '0'. The first place can be filled in only one way i.e. by '0'.

The remaining 6 places can be filled in $= \frac{6!}{2!\,3!}$

$= \frac{720}{2\times6} = 60$ ways.

$\therefore$ Numbers greater than one million $= 420 - 60$

$= 360$.

8. How many different words can be formed with the letters of the word BHARAT? In how many of these B and H are never together and how many of these begin with B and ends with T?

Sol:- There are 6 letters

B - 1 times

H - 1 times

A - 2 times

R - 1 time

T - 1 time

$\therefore$ No. of arrangements $= \frac{6!}{2!}$

$= 6\times5\times4\times3$

$= 360$

Now calculate the arrangements where B and H are together.

Arrangements with B and H together

$\boxed{BH}$ ARAT $= \frac{5!}{2!} = \frac{120}{2} = 60$

Since B & H can be arranged in 2ways $= 60 \times 2 = 120$ ways.

Now, calculate the arrangements where B and H are not together.

(Since B & H can be arranged in 2ways, $60 \times 2 = 120$)

Arrangements whose B and H are not together

= Total arrangements − Arrangements with B and H together.

$= 360 - 120$

$= 240$.

The number of arrangements of the letters in the word 'BHARAT' such that B and H never come together is $\underline{\underline{240}}$.

* Begin with B & ends with T :- B _ _ _ _ T $= \frac{4!}{2!} = 12$ ways

9. Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements:

(i) do the word start with P?

(ii) do all the vowels occur together?

(iii) do all the vowels not occur together?

(iv) do the ~~vowels~~ words begin with I and end in P?

(P.T.O)

Sol:- There are 12 letters in the word,
INDEPENDENCE.

$\therefore$ Total number of words $= \dfrac{(12)!}{(3!)(2!)(4!)}$

$= 1663200$.

(i) If the words starts with P, then we have 11 letters remaining.

$\therefore$ No. of words $= \dfrac{(11)!}{(3!)(2!)(4!)}$

$= 138600$

(ii) The vowels are I,E,E,E,E, consider it has single entity

$\boxed{IEEEE}$ NNNDDPC and this can be done in $\dfrac{8!}{(3!)(2!)}$ ways.

Corresponding to each of these arrangements, the 5 vowels can be arranged in $\dfrac{5!}{4!}$ ways.

$\therefore$ Total no. of words $= \dfrac{(8!)}{(3!)(2!)} \times \dfrac{5!}{4!}$

$= 16800$

(iii) No. of words with all vowels not occuring together

= Total no. of words − no. of words with all vowels occuring together

$= 1663200 - 16800$

$= 1646400$

(iv) If 1^{st} place is filled by I in 1 way, and the last place is filled by P in 1 way.
Then middle 10 places can be filled in

$$\frac{10!}{(3!)(2!)(4!)} = 12600 \text{ ways.}$$

$\therefore$ Total no. of words $= 1 \times 1 \times 12600 = 12600$.

10. Find the number of arrangements of the word PERMUTATION. In how many of these, 5 vowels occur together?

Sol: If E, U, A, I, O are consider as single entity Then

$\boxed{EUAIO}$ PRMTTN

$$\text{No. of arrangements} = \frac{7!}{(2!)} = 7\times6\times5\times4\times3$$
$$= 2520$$

$$\text{Arrangements of vowels within itself} = 5! = 120.$$

$$\therefore \text{Total no. of arrangements} = 2520 \times 120$$
$$= 3,02,400$$

Combination.

Combinations are selections made by taking some or all of a number of objects, irrespective of their arrangements.

The number of combinations of 'n' different things taken r at a time, denoted by ${}^nC_r$ and it is given by, ${}^nC_r = \frac{n!}{r!(n-r)!}$, where $0 \le r \le n$

i.e. ${}^nC_r = {}^nP_r / r!$

Important results on combinations

- The number of ways of selecting n objects out of n objects is :- ${}^nC_n = 1$
- The number of ways of selecting 0 objects out of n objects is :- ${}^nC_0 = 1$
- The number of ways of selecting 1 object out of n object is :- ${}^nC_1 = n$
- ${}^nC_r = {}^nC_{n-r}$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, $r \le n$
- Out of n objects, the number of ways of selecting 2 objects is ${}^nC_2 = \frac{n(n-1)}{2}$
- Clue words :- group, committee, sample, Selection, Subset.

We have, $^nC_r = \frac{n!}{(n-r)!\,r!} \rightarrow (1)$

(i) if $r = n$ in (1), we get

$$^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!\,n!}$$

$$\boxed{^nC_n = 1}$$

(ii) if $r = 0$ in (1), we get

$$^nC_0 = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!}$$

$$\boxed{^nC_0 = 1}$$

(iii) Put $r = n-r$ in (1), we get

$$^nC_{n-r} = \frac{n!}{(n-(n-r))!\,(n-r)!}$$

$$= \frac{n!}{(n-n+r)!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!}$$

$$\therefore \boxed{^nC_{n-r} = {^nC_r}}$$

Problems.

1. Evaluate each of the following.

(i) $^{10}C_8$ We have, $^nC_r = \frac{n!}{(n-r)!\,r!}$

$n = 10,\ r = 8$

Date :
Page No :

$$^{10}C_8 = \frac{10!}{(10-8)!\,8!}$$

$$= \frac{10!}{2!\,8!}$$

$$= \frac{10\times 9\times 8!}{(2\times 1)\,8!}$$

$$^{10}C_8 = 45.$$

(ii) $^{100}C_{98}$

$$^{100}C_{98} = \frac{(100!)}{(100-98)!\,98!}$$

$$= \frac{100\times 99\times \cancel{(98!)}}{(2!)\cancel{(98!)}}$$

$$= \frac{\cancel{100}^{50}\times 99}{\cancel{2}}$$

$$= 4950$$

2. If $^nC_{10} = {}^nC_{12}$, find $^{23}C_n$.

Sol:- Given, $^nC_{10} = {}^nC_{12}$

$$\frac{n!}{(n-10)!\,10!} = \frac{n!}{(n-12)!\,12!}$$

$$(n-12)!\,12! = (n-10)!\,10!$$

~~(n-12)(n-11)(n-10~~

$$\frac{(n-10)!}{(n-12)!} = \frac{12!}{10!}$$

$$\frac{(n-10)(n-11)(n-12)!}{(n-12)!} = \frac{12\times 11\times 10!}{10!}$$

$$(n-10)(n-11) = 12 \times 11$$

$$\Rightarrow n^2 - 11n - 10n + 110 = 132$$

$$\Rightarrow n^2 - 21n + 110 - 132 = 0$$

$$\Rightarrow n^2 - 21n - 22 = 0$$

$$\Rightarrow (n-22)(n+1) = 0$$

$$\Rightarrow n - 22 = 0$$

$$\boxed{n = 22}$$

-22

$-22 \quad 1$

-21

To find, $^{23}C_n$

$$^{23}C_{22} = \frac{(23)!}{(23-22)!(22)!}$$

$$= \frac{(23) \times (22)!}{1! \times (22)!}$$

$$^{23}C_{22} = \underline{\underline{23}}.$$

3. If $^8C_r - {}^7C_3 = {}^7C_2$, Find r.

Sol:- $^8C_r = {}^7C_2 + {}^7C_3$

$$^8C_r = \frac{7!}{(7-2)!\,2!} + \frac{7!}{(7-3)!\,3!}$$

$$^8C_r = \frac{8!}{(8-r)!\,r!} = \frac{7!}{5!\,2!} + \frac{7!}{4!\,3!}$$

$$= \frac{7 \times \cancel{6}^3}{\cancel{2}} + \frac{7 \times \cancel{6} \times 5}{\cancel{6}}$$

$$= 21 + 35$$

$$^8C_r = 56$$

(P.T.O)

For $r=0$, $^8C_0=1$

$r=1$, $^8C_1=8$

$r=2$, $^8C_2=28$

$\boxed{r=3,\ ^8C_3=56}$

$\therefore\ ^8C_3=56$, $r=3$

Other Possibilities are $^8C_{8-3}={}^8C_5=56$

[i.e $^nC_r={}^nC_{n-r}$]

$\therefore \boxed{r=3 \text{ or } r=5}$

4. If $^{n+2}C_8 : {}^{n-2}P_4 = 57:16$, find n

Sol:-

$$\frac{^{n+2}C_8}{^{n-2}P_4}=\frac{57}{16}$$

$$\Rightarrow \frac{\frac{(n+2)!}{(n+2-8)!\,8!}}{\frac{(n-2)!}{(n-2-4)!}}=\frac{57}{16}$$

$$\Rightarrow \frac{(n+2)!}{\cancel{(n-6)!}\,8!}\times\frac{\cancel{(n-6)!}}{(n-2)!}=\frac{57}{16}$$

$$\Rightarrow \frac{(n+2)(n+1)n(n-1)\cancel{(n-2)!}}{8!\,\cancel{(n-2)!}}=\frac{57}{16}$$

$$\Rightarrow (n+2)(n+1)n(n-1)=\frac{57\times 8!}{16}$$

Date :
Page No :

$(n+2)(n+1)n(n-1) = 1,43,640$

By Inspection, $n = 19$

$(21)(20)(19)(18) = 1,43,640.$

5. If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshake happen in the party.

Sol:- 2 person are to be chosen from 12 person to have a handshake.

$\therefore$ Possible ways, ${}^{12}C_2 = \frac{12!}{10! \times 2!}$

$= \frac{\cancel{12}^{6} \times 11 \times \cancel{10!}}{\cancel{10!} \times (\cancel{2} \times 1)}$

$= \underline{\underline{66}}.$

6. In how many ways a committee of 5 members can be selected from 6 men and 5 women, consisting of 3 men and 2 women?

Total no. of ways of selecting 3 men from 6 men is ${}^6C_3$

Sol:- Total no. of ways of selecting 2 women from 5 women is ${}^5C_2$

Total no. of ways of selecting,

$${}^6C_3 \times {}^5C_2 = \frac{6!}{3!\,3!} \times \frac{5!}{3!\,2!}$$

$$= \frac{\cancel{6} \times 5 \times 4 \times \cancel{3!}}{(\cancel{3} \times \cancel{2}) \times \cancel{3!}} \times \frac{5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{3!} \times \cancel{2!}}$$

$$= 20 \times 10$$

$$= 200 \text{ ways}.$$

7. How many triangles can be obtained by joining 12 points, wherein five of which are collinear?

$n=12,\ r=3$

① Total combinations $= {}^{12}C_3 = \frac{12!}{3!(12-3)!}$

$= \frac{12!}{3!\,9!}$

$= \frac{\cancel{12}^{2}\times 11\times 10}{\cancel{3}\times\cancel{2}}$

$= 220$

② Calculating Collinear combinations

Collinear combinations $= {}^{5}C_3 = \frac{5!}{3!\,2!}$

$= \frac{5\times\cancel{4}^{2}}{\cancel{2}}$

$= 10$

③ No. of valid triangles = Total Combinations − Collinear combinations

$= 220 - 10$

$= 210.$

Date :
Page No :

8. What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these .
(i) Four cards are of same suit.
(ii) Four cards belongs to different suit.
(iii) Two are red cards and two are black cards.
(iv) Cards are of same colour.

Sol:

There are 52 playing cards, out of which 4 cards to be choosen.

$\therefore\ ^{52}C_4$ is the required no. of ways

$$= \frac{52!}{48!\,4!} = \frac{52 \times 51 \times 50 \times 49 \times 48!}{48! \times 4 \times 3 \times 2 \times 1}$$

$$= 270725 .$$

(i) Four cards are of same suit.

No. of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4\,{}^{13}C_4$

$$= 4 \times {}^{13}C_4$$

$$= 4 \times \frac{13!}{9!\,4!}$$

$$= 2860$$

(ii) Four cards belongs to different suit i.e., we have to choose one card from each suit.

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13 \times 13 \times 13 \times 13$$

$$= 13^4$$

(iii) Out of 52 cards, 26 are red cards and 26 are black cards. We have to choose 2 cards from black and 2 cards from red.

$$\therefore\ {}^{26}C_2 \times {}^{26}C_2 = \left({}^{26}C_2\right)^2 = 105625$$

Date :
Page No :

(iv) Cards should be of same colour. 4 red cards can be selected out of 26 red cards or 4 black cards can be selected out of 26 black cards.

$\therefore$ The required no. of ways $= {}^{26}C_4 + {}^{26}C_4$

$= 2 \cdot {}^{26}C_4$

$= 2(14950)$

$= 29,900$

9 A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of:
(i) exactly 3 girls (ii) at least 3 girls
(iii) at most 3 girls.

Sol:- A committee of 7 has to be formed from 9 boys and 4 girls.

(i) No. of ways of selecting exactly 3 girls to form a committee $= {}^4C_3 \times {}^9C_4$

$= \frac{4 \times \cancel{3} \times \cancel{2}}{\cancel{3} \times \cancel{2} \times 1} \times \frac{9 \times \cancel{8}^{2} \times 7 \times \cancel{6}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$

$= 4 \times 9 \times 2 \times 7$

$= 504$

(ii) No. of ways of selecting atleast 3 girls $= {}^4C_3 \times {}^9C_4 + {}^4C_4 \times {}^9C_3$

$= 504 + \frac{\cancel{9}^{3} \times \cancel{8}^{4} \times 7}{\cancel{3} \times \cancel{2} \times 1}$

$= 504 + 84 = 588.$

Date :
Page No :

(iii) No. of selecting almost 3 girls $= {}^4C_3 \times {}^9C_4 + {}^4C_2 \times {}^9C_5 + {}^4C_1 \times {}^9C_6 + {}^4C_0 \times {}^9C_7$.

$= 504 + 756 + 336 + 36$

$= 1632$.

10. In how many ways can a cricket team of eleven be chosen out of a batch of 15 players if,
(i) there is no restriction on the selection
(ii) a particular player is always choosen.
(iii) a particular player is never choosen.

Sol:- i) No. of ways of choosing 11 players from 15 is ${}^{15}C_{11} = {}^{15}C_4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 15 \times 7 \times 13 = 1365$.

ii) If a particular player is always chosen there will be only 14 players left; in which 10 are to be selected.

$${}^{14}C_{10} = {}^{14}C_4 = \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1} = 1001 \text{ ways}$$

(iii) If a particular player is never chosen. we have to select 11 players out of remaining 14 players in ${}^{14}C_{11}$ ways.

i.e. $${}^{14}C_3 \text{ ways} = \frac{14 \times 13 \times 12}{3 \times 2 \times 1} = 364 \text{ ways}$$

Date :
Page No :

(5M) State and prove Pigeonhole principle.

Statement:- If n pigeons are assigned to m pigeonholes and $m<n$, then some pigeonhole contains atleast two pigeons.

Proof:-

Let $h_1, h_2, \ldots, h_m$ denote 'm' pigeonholes and

$P_1, P_2, \ldots, P_m, P_{m+1}, \ldots P_n$ denote 'n' pigeons (where $m<n$)

We consider the assignment of the 'n' pigeons to 'm' pigeonholes as follows:-

Assign pigeon P_1 to the pigeonhole h_1, pigeon P_2 to the pigeonhole $h_2, \ldots$ and pigeon P_m to the pigeonhole h_m.

This assigns as many pigeons possible to individual pigeonholes. Since $m<n$, there are P_i pigeons $(i=n-m)$ that have not yet been assigned.

$\therefore$ Atleast one pigeonhole will be assigned a second pigeon.

(5M). Explain principles of Counting.

A:- 1. Sum Rule principle:-

The sum rule states that 'If some event A can occur in 'm' ways and a second event B can occur in 'n' ways, and suppose both events cannot occur simultaneously. Then

A or B can occur in m+n ways'.

In General, Suppose an event A_1 can occur in n_1 ways, a second event A_2 can occur in n_2 ways, and following A_2, a third event A_3 can occur in n_3 ways and so on. If no two events can occur at the same time, then one of the events can occur in,

$$n_1+n_2+n_3+\ldots+\text{ ways.}$$

2. Product Rule principle:-

The product rule states that,' If there is an event A which can occur in 'm' ways and, independent of this event, there is a second event B which can occur in 'n' ways. Then combinations of A and B can occur in $m \times n$ ways.

In General, the above principle can be extended to three or more events. That is, suppose an event A_1 can occur in n_1 ways, a second event A_2 can occur in n_2 ways, and following A_2; a third event A_3 can occur in n_3 ways, and so on. If the events occur one after the other, then all the events can occur in

$$n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot \text{ ways.}$$