Subject

Date	Title	Page No	Teacher's Sign
	Set Theory.		
		Completed	
	→ Fundamentals of Set theory	✓	
	→ Set Operations	✓	
	→ Laws of Set Theory	✓	
	→ Counting and Venn diagrams	✓	
	→ Cartesian product	✓	
	→ Relations, Types of relations	✓	
	→ Functions, Types of functions		
	→ Function composition		
	→ Inverse function		
	→ Mathematical induction.		

Discrete structure

Discrete Mathematics is the branch of mathematics dealing with objects that can assume only distinct, separated values.

Applications of discrete mathematics.
- Computer algorithms
- Programming languages
- Cryptography
- Software development
- Google maps [To find the shortest distance b/w two points]

[Discrete mathematics underpins cryptography through the use of algorithms and structures like graphs, finite fields, and combinatorics, which helps secure data encryption, decryption and the authenciation within cryptographic systems].

Fundamentals of set theory.

The foundation of set theory was laid by German mathematician, George Cantor. He is regarded as the father of set theory

Definition :- A set is an unordered well-defined collection of objects.

- The objects in a set are called the elements or members of the set.

Cardinality :- The number of elements of a finite set A is called cardinality and denoted by $n(A)$ or $|A|$

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Notation :-

When x is an element of a set A, we write $x \in A$, read as 'x belongs to A'

When x is not an element of a set A, we write $x \notin A$, read as 'x does not belongs to A'.

example :- $A = \{1, 2, 3, 4\}$
Here $3 \in A$ but $5 \notin A$.

Representation of sets.
A set can be represented in two ways:-
1. Roaster form or tabular form.
2. Set builder form or Rule form. 1.

Roaster form :- In this method, all the elements of a set are listed, the elements are being separated by commas, and enclosed within a pair of brackets { }.
ex:- A set of natural numbers less than 5.
$A = \{1, 2, 3, 4\}$

Rule method :- In this method, we specify the set by stating a characteristic property which all the elements of the set possess. 2.
ex:- $A = \{x \mid x$ is a positive integer less than $4\}$
or $A = \{x \mid x \in Z^+$ and $x < 4\}$
"The set A of all objects 'x' such that 'x' is a positive integer less than.
The vertical bar | is read as "Such that" 3

Standard notations of sets.

ϕ → empty set or null set
U → Universal set
N → Set of all natural numbers
Z → Set of integers
Z^+ → Set of positive integers
Z^- → Set of negative integers
Q → Set of rational numbers
R → The set of real numbers
C → The set of complex numbers.

Types of sets.

1. Null set or Empty set or Void set

A set which does not contain any element is called the null set or the empty set or void set. It is denoted by ϕ or $\{\}$.

Note:- $A = \{0\}$ is not a null set

ex:- $A = \{$ The set of all natural number less than 1 $\}$.

$B = \{x : x^2 = 4, x \text{ is odd}\}$

2. Finite set : A set containing countable number of elements is called finite set.

ex:- $V = \{a, e, i, o, u\}$

$A = \{x \mid x \in N, 3 < x < 9\}$

less than 4"

3 Infinite set:- A set containing uncountable number of elements is called infinite set.

ex:- A = The set of Natural numbers.

$B = \{x \mid x \in N,\ x \text{ is a multiple of } 2\}$

4. Singleton set : A set containing only one element is known as singleton set.

ex :- $B = \{X$ is an even prime number$\}$

$B = \{2\}$

$C = \{x \mid x \in N,\ x^2 = 25\}$

5. Equal sets :- Two sets A and B are said to be equal if they have exactly the same elements.

Symbolically written as $A = B \Leftrightarrow \{x \in A \text{ iff } x \in B\}$

ex:- $A = \{x : x$ is a letter in the word 'flow'$\}$

$B = \{x : x$ is a letter in the word 'wolf'$\}$

Then $A = B$ as A and B have the same elements.

6. Equivalent sets :- Two finite sets A and B are said to be equivalent if they have same number of elements.

ex:- Let $A = \{a, e, i, o, u\}$

$B = \{1, 2, 3, 4, 5\}$

$n(A) = 5$ and $n(B) = 5$

$\Rightarrow$ The sets A and B are equivalent.

7. Subset :- A set A is said to be subset of B, if every element of set A is also the element of set B.

Symbolically, $A \subseteq B \Leftrightarrow \{x \mid x \in A \Rightarrow x \in B\}$

ex:- $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$ Then $A \subseteq B$

If A is not a subset of set B iff there is atleast one element of A that is not an element of B.
Symbolically. $A \not\subseteq B \leftrightarrow \{x \mid x \in A \Rightarrow x \notin B\}$

Note:- ϕ and given set always itself a subsets.

8. Super set :- If A and B are any two sets, then B is called a super set of A iff $A \subset B$ but $A \neq B$.
[can be written as $B \supset A$]

ex:- $A = \{1, 2, 3\}$
$B = \{1, 2, 3, 4, 5\}$
$n(A) = 3$, $n(B) = 5$
$\therefore A \subset B$ but $A \neq B$
$\Rightarrow$ B is a Superset of A.

9. Propersubset :- If A and B are the two sets. Then, A is a proper subset of B if every element of A is in B but there is atleast one element of B that is not in A and is denoted by $A \subset B$.
ex:- $A = \{2, 3, 4\}$, $B = \{2, 3, 4, 5\}$,
Then A is a proper subset of B.

10. Universal set :- It is a basic set under which all other sets are considered to be its subsets. It is usually denoted by U.
ex:- For the set of integers (Z), the universal set can be set of real numbers (R).

** 11. Power set :- Let A be any set, then power set of A is denoted by P(A). It is a set containing all the possible subsets of set A.
$n(P(A)) = 2^n$, where n is the no. of elements of the set A.

ex:- $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$\therefore n(P(A)) = 2^2 = 4$.

12. Disjoint sets :- Two sets are said to be disjoint if they have no element in common.

ex:- The sets, $A = \{0, 4, 7, 9\}$ and $B = \{3, 6, 10\}$ are disjoint.

13. Comparable sets : Two sets A and B are said to be comparable if $A \subseteq B$ or $B \subseteq A$.

14. Not comparable set :- Two sets A and B are said to be not comparable if $A \not\subseteq B$ and $B \not\subseteq A$.

15. Multiset :- A multiset is unordered collection of elements where the repetition of elements matters.

ex:- $A = \{1, 1, 2, 3, 4, 4\}$

Operations on sets :-

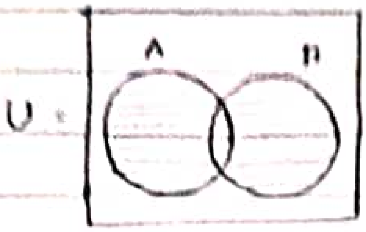
1. Union of sets

Let A and B are any two sets. Then the union of A and B is denoted by $A \cup B$ and is defined to be the set of all those elements, which are in A or in B or in both.

Thus, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Note :- A diagrammatic or graphical representation of a set is called Venn diagram.

The venn diagram for $A \cup B$

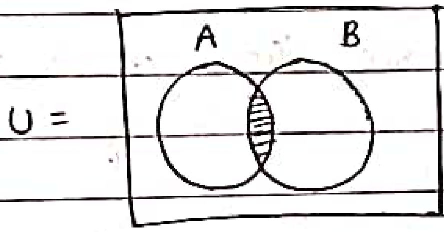


2. <u>Intersection of sets.</u>

Let A and B be any two sets. Then the intersection of A and B is denoted by $A \cap B$ is defined to be the ~~difference~~ set of all <u>common elements</u> between A and B.

Thus, $A \cap B = \{x / x \in A \text{ and } x \in B\}$

The venn diagram for $A \cap B$.



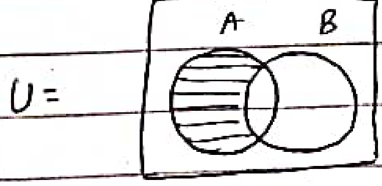
3. <u>Difference of two sets.</u>

Let A and B are any two sets. Then the difference of A and B is denoted by $A - B$ is a set containing all the elements of A which are not in B.

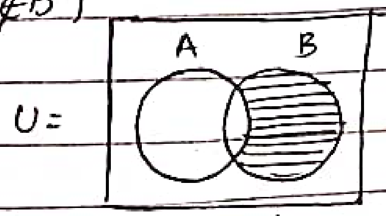
Similarly, $B - A$ is set containing all the elements of B which are not in A.

Thus, $B - A = \{x / x \in B \text{ and } x \notin A\}$

$A - B = \{x / x \in A \text{ and } x \notin B\}$



A - B



B - A

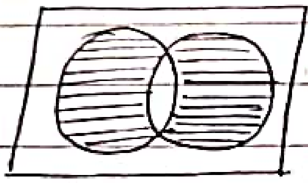
4. <u>Symmetric difference</u>

The symmetric difference of any two sets A and B is the set containing the elements that are in A or in B, but not in both and is denoted by $A \Delta B$.

i.e. $A \Delta B = (A \cup B) - (A \cap B) = \{x \mid x \in A \cup B \text{ and } x \notin A \cap B\}$

it is same as $(A-B) \cup (B-A)$

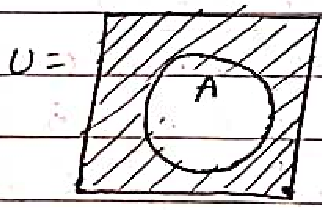
The Venn diagram for $A \Delta B$ is shown,



5. <u>Complement of a set</u>

Let A be any set, then A^c is the set which consists of all the elements of universal set which are not in A.

$A^c = U - A$.



Laws of set theory [Algebra of sets]

1. Commutative law :- If A and B are any two sets. Then,
(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$.

2. Associative law :- If A, B and C are any three sets, Then,
(i) $A \cup (B \cup C) = (A \cup B) \cup C$
(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

3. Distributive law :- If A, B, C are any three sets. Then
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. De Morgan's law :- If A and B are any two sets then
(i) $(A \cup B)' = A' \cap B'$
(ii) $(A \cap B)' = A' \cup B'$

5. Idempotent law :- Let A be any set. Then,
(i) $A \cup A = A$ (ii) $A \cap A = A$.

6. Identity Law :- Let A be any set, Then,
(i) $A \cup \phi = A$ (ii) $A \cap U = A$.

7. Double complement law :- Let A be any set, Then,
$\bar{\bar{A}}$ or $(A')' = A$

8. Inverse law :- Let A be any set,
Then (i) $(A \cup \bar{A}) = U$ $A \cup \bar{A} = U$
(ii) $(A \cap \bar{A}) = \phi$ $A \cap \bar{A} = \phi$.

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9. Absorption law :- Let A and B are any two sets.
Then, (i) $A \cap (A \cup B) = A$
(ii) $A \cup (A \cap B) = A$.

Counting principle

1. Cardinality of a Finite set :-

If A is finite set with n-distinct elements, then n is called the cardinality of A. The cardinality of A is denoted by $|A|$ [or $n(A)$]

2. Cardinality of union of two sets :-
If A and B are any two finite sets then, the number of elements in $A \cup B$,
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

3. If $A \cap B = \phi$ and if A and B are any two finite sets. Then $n(A \cup B) = n(A) + n(B)$

4. If A, B and C are any three finite non-disjoint-sets, Then,
$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

5. If A and A' both are finite sets, then $n(A') = n(U) - n$
where U is the universal set.

6. If A and B are any two finite sets. Then,
$$n(A - B) = n(A) - n(A \cap B)$$
$$n(B - A) = n(B) - n(A \cap B)$$

Proof of De Morgan's law

$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

(i) Let $x \in \overline{(A \cup B)} \Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in \bar{A}$ and $x \in \bar{B}$

$\Rightarrow x \in (\bar{A} \cap \bar{B})$

$\Rightarrow \overline{(A \cup B)} \subseteq \bar{A} \cap \bar{B}$

(ii) Let $x \in (\bar{A} \cap \bar{B})$

$\Rightarrow x \in \bar{A}$ and $x \in \bar{B}$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \in \overline{(A \cup B)}$

$\Rightarrow (\bar{A} \cap \bar{B}) \subseteq \overline{(A \cup B)}$

From (i) & (ii)

$\Rightarrow \overline{(A \cup B)} = \bar{A} \cap \bar{B}$

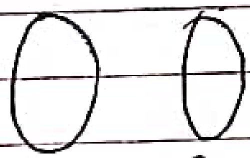
Similarly, we get proof of $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.

Venn diagrams.

A diagrammatic or graphical representation of a set is called Venn diagram.

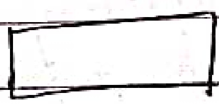
In Venn diagram usually sets are represented by circles or a closed figure and universal set is represented by a rectangle.

1. A set is represented by a circle or a closed figure



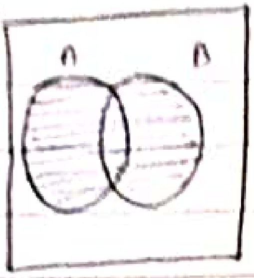
A B

2. Universal set

U = 

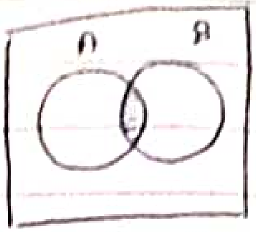
3. $A \cup B$

$U =$



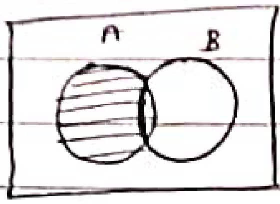
4. $A \cap B$

$U =$

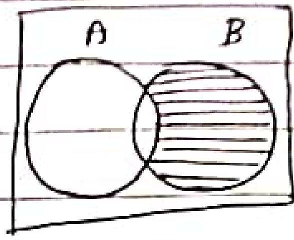


5. $A - B$

$U =$

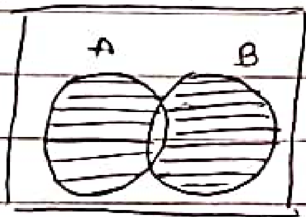


6. $B - A$



7. $A \Delta B = (A - B) \cup (B - A)$

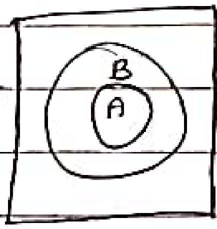
or $A \Delta B = (A \cup B) - (A \cap B)$



Shaded portion is $A \Delta B$

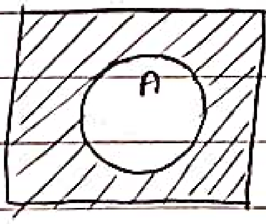
8. Subset $A \subseteq B$

$U =$



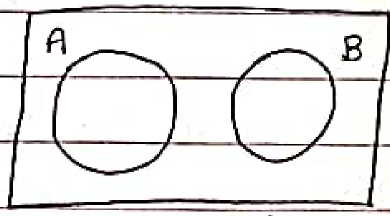
9. A' or $U - A$

$U =$



10. Disjoint sets $(A \cap B = \phi)$

$U =$



*** Principle of duality states that any established result involving sets & complements & operations of union & intersection gives a corresponding dual result by replacing $\cup$ by $\cap$, ϕ by U & $\cup$ by $\cap$, & vice versa.

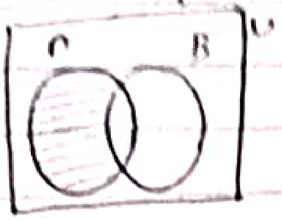
ex- Consider, $A \cap \bar{A} = \phi$, Applying Principle of duality.

$A \cup \bar{A} = U$

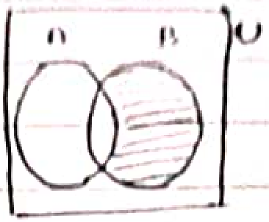
4) $A - B = \{1, 2, 4\}$

$B - A = \{7, 8\}$

$A \cup B = \{1, 2, 4, 5, 7, 8, 9\}$, $A = ?$ $B = ?$



A - B



B - A

$$A = (A \cup B) - (B - A)$$
$$= \{1, 2, 4, 5, 7, 8, 9\} - \{7, 8\}$$
$$= \{1, 2, 4, 5, 9\}$$

$$B = (A \cup B) - (A - B)$$
$$= \{1, 2, 4, 5, 7, 8, 9\} - \{1, 2, 4\}$$
$$= \{5, 7, 8, 9\}$$

5] Union of sets is commutative.
i.e. if A & B are any sets, then prove
$A \cup B = B \cup A$.

Let $x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in (B \cup A)$

Hence $(A \cup B) \subseteq (B \cup A) \rightarrow (1)$

Conversely,

Let $x \in B \cup A$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in (A \cup B)$

Thus $(B \cup A) \subseteq (A \cup B) \rightarrow (2)$

from (1) & (2)

$A \cup B = B \cup A$.

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Q: Union of sets is associative, i.e. If A, B and C are any three sets, then prove $(A \cup B) \cup C = A \cup (B \cup C)$

Let $x \in (A \cup B) \cup C$

$\Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$

Hence $(A \cup B) \cup C \subseteq A \cup (B \cup C) \rightarrow$ (1)

Conversely, $x \in A \cup (B \cup C)$

$\Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in A \cup B$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cup C$

$\Rightarrow A \cup (B \cup C) \subseteq (A \cup B) \cup C \rightarrow$ (2)

From (1) & (2)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Q:- Prove that, for any three sets A, B, C,

$$\overline{(A \cap B) \cup C} = (\bar{A} \cup \bar{B}) \cap \bar{C}$$

Soln. Let $A \cap B = D$

$$\therefore \overline{(A \cap B) \cup C} = \overline{D \cup C}$$
$$= \bar{D} \cap \bar{C} \quad \text{[By Demorgan's law]}$$
$$= \overline{(A \cap B)} \cap \bar{C}$$
$$= (\bar{A} \cup \bar{B}) \cap \bar{C} \quad \text{[By Demorgan's law]}$$
$$\overline{(A \cap B) \cup C} = (\bar{A} \cup \bar{B}) \cap \bar{C}$$

This proves the result.

Q:- Explain Principle of counting

→ It is a mathematical rule that helps you calculate the total number of possible outcomes when multiple events occur simultaneously.

& intersection.

Ex:

1\. Find the union & of each of the following pairs of sets.

i) $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7\}$

$A \cup B = \{1,2,3,4,5,6,7\}$

$A \cap B = \{3,4\}$

ii) $X = \{x \mid x \in Z^+ \text{ and } x^2 > 7\}$; $Y = \{1,2,3\}$

$X = \{2,3,4,5,\ldots\}$, $Y = \{1,2,3\}$

$X \cup Y = \{1,2,3,4,5,\ldots\}$

$X \cap Y = \{2,3\}$

iii) $A = \{x \mid x \in N \text{ and } 1 < x \leq 5\}$

$B = \{x \mid x \in N \text{ and } 5 < x < 10\}$

$A = \{2,3,4,5\}$ & $B = \{6,7,8,9\}$

$A \cup B = \{2,3,4,5,6,7,8,9\}$

$A \cap B = \{\ \}$

2 if $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$,
$C = \{7,8,9,10,11\}$, $D = \{10,11,12,13,14\}$

Find (i) $A \cup B \cup C$

$= \{1,2,3,4,5,6,7,8,9,10,11\}$

(ii) $A \cup C \cup D$

$= \{1,2,3,4,5,7,8,9,10,11,12,13,14\}$

3. Let if $A = \{1,2,3,4\}$, $B = \{4,6,7,8\}$, $C = \{2,4,6,8\}$

Verify the following identities

i) $(A \cup B) \cup C = A \cup (B \cup C)$

LHS: $(A \cup B) \cup C = \{1,2,3,4,6,7,8\} \cup \{2,4,6,8\}$
$= \{1,2,3,4,6,7,8\}$

RHS: $A \cup (B \cup C) = \{1,2,3,4\} \cup \{2,4,6,7,8\}$
$= \{1,2,3,4,6,7,8\}$

LHS = RHS, Hence verified.

ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

LHS: $A \cup (B \cap C) = \{1,2,3,4\} \cup \{4,6,8\}$
$= \{1,2,3,4,6,8\}$

RHS: $(A \cup B) \cap (A \cup C) = \{1,2,3,4,6,7,8\} \cap \{1,2,3,4,6,8\}$
$= \{1,2,3,4,6,8\}$

LHS = RHS, Hence verified

4. If $A = \{2,4,6,8,10,12\}$, $B = \{3,4,5,6,7,8,10\}$ find

i) $A - B = \{2,4,6,8,10,12\} - \{3,4,5,6,7,8,10\}$
$= \{2,12\}$

ii) $B - A = \{3,4,5,6,7,8,10\} - \{2,4,6,8,10,12\}$
$= \{3,5,7\}$

iii) $(A-B) \cup (B-A) = \{2,12\} \cup \{3,5,7\}$
$= \{2,3,5,7,12\}$

5. If $A = \{a,b,c,d,e\}$, $B = \{a,c,e,g\}$; $C = \{b,e,f,$

find

i] $A \cap (B-C) = \{a,b,c,d,e\} \cap \{a,c\}$
$= \{a,c\}$

ii] $A - (B \cup C) = \{a,b,c,d,e\} - \{a,b,c,e,f,g\}$
$= \{d\}$

iii] $A - (B \cap C) = \{a,b,c,d,e\} - \{e,g\}$
$= \{a,b,c,d\}$

6. Let $U = \{1,2,3,4,5,6,7,8,9,10\}$
$A = \{1,3,5,7\}$, $B = \{2,4,6,8\}$, $C = \{4,5,6,7\}$

Find

i) $A' = U - A$
$= \{1,2,3,4,5,6,7,8,9,10\} - \{1,3,5,7\}$
$= \{2,4,6,8,9,10\}$

ii) $B' = U - B$
$= \{1,3,5,7,9,10\}$

iii) $C' = U - C$
$= \{1,2,3,8,9,10\}$

iv) $(A \cup C)' = U - (A \cup C)$
$= \{1,2,3,4,5,6,7,8,9,10\} - \{1,3,4,5,6,7\}$
$= \{2,8,9,10\}$

v) $(A')' = U - A'$

$= \{1,2,3,4,5,6,7,8,9,10\} - \{2,4,6,8,9,10\}$

$= \{1,3,5,7\} = A.$

7. If A and B are two sets such that $n(A) = 25$, $n(B) = 37$ and $n(A \cup B) = 50$ find $n(A \cap B)$

Sol: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 25 + 37 - 50$

$= 62 - 50$

Hence, $n(A \cap B) = 12$

8. In a committee, 60 people speak Hindi, 30 speak English and 20 speak both Hindi and English. How many speak atleast one of these two languages.

Sol: Let H be Set of all people who speak Hindi

E = set of all people who speak English

Given $n(H) = 60$ and $n(E) = 30$

$n(H \cap E) = 20$

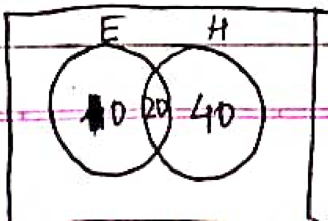
$n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$= 60 + 30 - 20$

$= 90 - 20$

$= 70$

∴ 70 people speak at least one of these two languages.



9. In an examination 56% of the candidates failed in Mathematics and 48% in Physics. If 18% failed in both Mathematics and physics, find the percentage of those who passed in both the subjects.

Sol:- Let Total candidates = 100

No. of candidates failed in Mathematics $n(M) = 56$

No. of candidates failed in Physics $= n(P) = 48$

No. of candidates failed in both the subjects $= n(M \cap P) = 18$

No. of candidates failed $= n(M \cup P)$

WKT, $n(M \cup P) = n(M) + n(P) - n(M \cap P)$

$= 56 + 48 - 18$

$= 104 - 18$

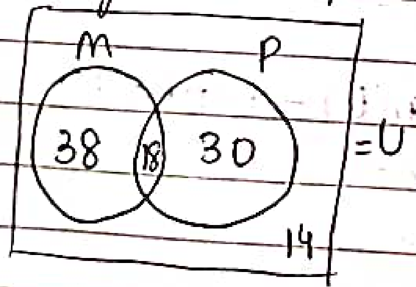
$= 86$,

Now, No. of passed candidates = Total candidates $- n(M \cup P)$

$= 100 - 86$

$= 14$

∴ Percentage of candidates who passed both the subjects $= 14\%$



PYP

10\. In a group of 80 people, 42 like coffee, 60 like tea & each person like atleast one of the two drinks. find how many people like both coffee and tea?

Sol:- Given,

$n(U) = 80$

$n(C) = 42$

$n(T) = 60$

$n(C \cap T) = ?$

$n(C \cup T) = n(C) + n(T) - n(C \cap T)$

$\Rightarrow n(C \cap T) = n(C) + n(T) - n(C \cup T)$

$= 42 + 60 - 80$

$= 102 - 80$

$= 22$

$\therefore$, 22 people like both coffee & tea

PYP

11\. Suppose that in a certain examination, 200 students appear for mathematics, 50 appear for physics, 100 appear for chemistry, 20 appear for mathematics and physics, 60 appear for mathematics and chemistry, 35 appear for physics and chemistry, while 245 appear for mathematics or physics or Chemistry. Determine the number of students appear

i] for all the three subjects

ii] Exactly for one of the subjects.

Also construct the venn diagram.

Sol:- $n(M) = 200$

$n(P) = 50$

$n(C) = 100$

$n(M \cap P) = 20$

$n(M \cap C) = 60$

$n(P \cap C) = 35$

$n(P \cup M \cup C) = 245$

$n(P \cap M \cap C) = ?$

$$n(P \cup M \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(P \cap M \cap C)$$

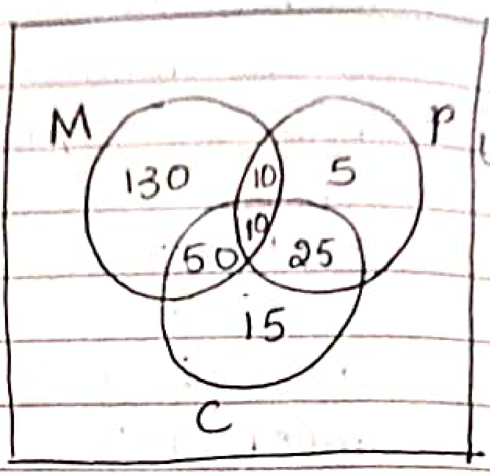
$$245 = 200 + 50 + 100 - 20 - 60 - 35 + n(P \cap M \cap C)$$

$$245 = 350 - 115 + n(P \cap M \cap C)$$

$$245 = 235 + n(P \cap M \cap C)$$

$n(P \cap M \cap C) = 245 - 235 = 10$

No. of students who appear for all the 3 subjects is 10.



ii] Now, we have to find no of students who appear exactly for one of the subjects
$= 130 + 5 + 15$
$= 150$

∴ No. of students who appear exactly for one of the subject = 150.

12. In a group of 65 people, 40 like cricket, 10 like both cricket and Tennis. How many like Tennis only and not cricket? How many like Tennis?

Sol:
$n(C \cup T) = 65$
$n(C) = 40$
$n(C \cap T) = 10$
$n(T) = ?$

$$n(C \cup T) = n(C) + n(T) - n(C \cap T)$$
$$65 = 40 + n(T) - 10$$
$$n(T) = 65 - 40 + 10$$
$$n(T) = 75 - 40$$
$$n(T) = 35$$

∴ 35 people like tennis ~~only and not cricket~~.

To find the total number of people who like tennis only not cricket.

~~People who like tennis = $n(T) + n(T \cap C)$~~
~~$= 35 + 10$~~
~~$= 45$~~

~~So, 45 people like tennis in total.~~

Finding the no. of people who like tennis only & not cricket.

$\Rightarrow n(T-C) = n(T) - n(C \cap T)$
$= 35 - 10$
$= 25$

$\therefore$ 35 people like Tennis and 25 people like tennis only and not cricket.

13. In a survey of 400 Students in a school, 110 were listed as taking apple juice, 140 as taking orange juice and 85 were listed as taking both apple as well as Orange juice. Find how many students were taking neither Apple juice nor orange juice.

$n(U) = 400$

Sol:- $n(A) = 110$
$n(O) = 140$
$n(A \cap O) = 85$
$n(A' \cap O') = ?$

$n(A \cup O) = n(A) + n(O) - n(A \cap O)$
$= 110 + 140 - 85$
$= 250 - 85$
$n(A \cup O) = 165$

$n(A' \cap O') = n(A \cup O)'$
$= U - n(A \cup O)$
$= 400 - 165$
$= 235$

14. In a survey it was found that 21 persons liked product P_1, 26 liked product P_2 and 29 liked product P_3. If 14 persons liked product P_1 and P_2 ; 12 persons liked product P_3 and P_1 ; 14 persons liked products P_2 and P_3 and 8 liked all the three products. Find how many liked product P_3 only.

$n(P_1) = 21$

$n(P_2) = 26$

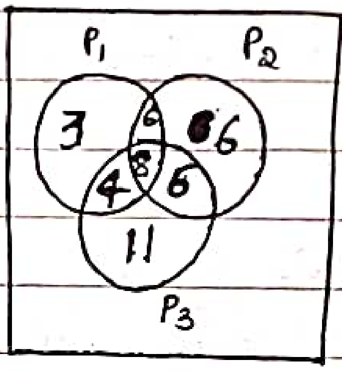
$n(P_3) = 29$

$n(P_1 \cap P_2) = 14$

$n(P_2 \cap P_3) = 14$

$n(P_3 \cap P_1) = 12$

$n(P_1 \cap P_2 \cap P_3) = 8$



P_3 only $= n(P_3) - n(P_2 \cap P_3) - n(P_3 \cap P_1) + n(P_1 \cap P_2 \cap P_3)$

$= 29 - 12 - 14 + 8$

$= 17 - 14 + 8$

$= 3 + 8$

$= 11$

Relations.

Cartesian product :-
Let A and B be two non-empty sets. Then the cartesian product of A and B is the set denoted by $(A \times B)$ consisting of all ordered pairs (a,b) such that $a \in A$ and $b \in B$.

$$\therefore A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

Ordered pair :- Two elements a and b listed in a specific order and enclosed in parentheses form an ordered pair (a,b)

Note :- ① By interchanging the positions of the components, the ordered pair is changed. Thus $(a,b) \neq (b,a)$

② If $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$

example 1 :-

1. If $\left(\frac{x}{3}+1, y-\frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$ find x and y

Sol: Given $\left(\frac{x}{3}+1, y-\frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$

$\Rightarrow \frac{x}{3}+1 = \frac{5}{3}$ and $y - \frac{2}{3} = \frac{1}{3}$

$\Rightarrow \frac{x}{3} = \frac{5}{3} - 1$ and $y = \frac{1}{3} + \frac{2}{3}$

$\Rightarrow \frac{x}{3} = \frac{2}{3}$ and $y = \frac{3}{3}$

$\Rightarrow x = 2$ and $y = 1$

2:- Let $A = \{1, 2\}$, $B = \{3, 4, 5\}$ find $A \times B$

$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

Ordered triplet :- If A, B, C are three sets then (a, b, c) where $a \in A$, $b \in B$ and $c \in C$ is called ordered triplet.

Cartesian product of three sets :-

If A, B and C are three sets, then
$A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$

3:- If $A = \{1, 0\}$, $B = \{2, -2\}$, $C = \{0, -1\}$, find $A \times B \times C$

$A \times B \times C = \{(1,2,0), (1,2,-1), (1,-2,0), (1,-2,-1),$
$(0,2,0), (0,2,-1), (0,-2,0), (0,-2,-1)\}$

Relations

Let A and B be two non-empty sets. Then a relation R from A to B is a subset of $A \times B$.

Thus R is a relation from A to $B \Leftrightarrow R \subseteq (A \times B)$.

if $(a, b) \in R$, then we say that 'a is related to b' and we write aRb

if $(a, b) \notin R$, then 'a is not related to b' and we write $a \not R b$.

Inverse of a relation :- Let A, B be two non-empty sets. Let R be a relation from A to B. The inverse relation of R denoted by R^{-1} is a relation

Note:- If A is a set with m elements and B is a set with n elements then, then total no. of relations from A to B $= 2^{mn}$.

Domain of the relation R: The set of all first co-ordinates of elements of R is called the 'domain of R' denoted by dom(R).

Thus $dom(R) = \{a \mid (a,b) \in R\}$

Range of the relation R: The set of all second co-ordinates of elements of R is called the 'range of R' denoted by range(R)

Thus $range(R) = \{b \mid (a,b) \in R\}$

Co-domain of the relation R:

The set B is called the co-domain of R.

Range $\subseteq$ co-domain.

Representation of Relations

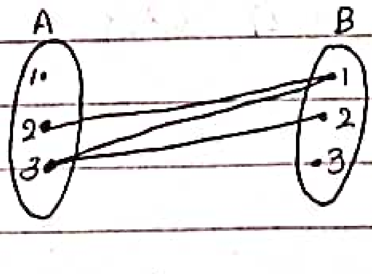
(i) By language (Set-builder) (ii) By ordered pairs (Roaster)
(iii) By arrow form (iv) By matrix form
(v) By co-ordinates (vi) By graph form.

Suppose $A = \{1,2,3\}$ then

(i) $R = \{(a,b) \mid a > b \text{ and } \forall a, b \in A\}$

(ii) $R = \{(2,1), (3,1), (3,2)\}$

(iii)



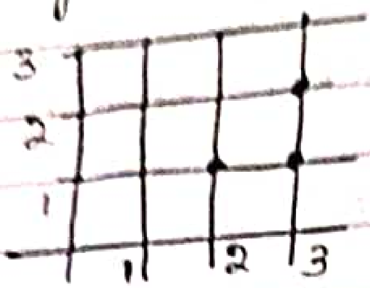
(iv)

	1	2	3
1	0	0	0
2	1	0	0
3	1	1	0

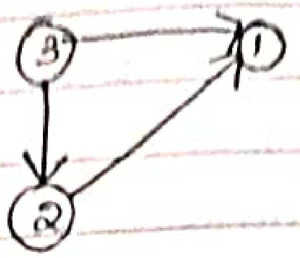
$$M_R = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

from B to A defined by $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

v) By Co-ordinates



vi) By graph form



Kinds of relation.

1. Empty relation: A relation R on a set A is said to be an empty relation if no element of set A is related to any other element of set A.

$R = \phi \subset A \times A$.

ex:- Let $A = \{1, 2, 3\}$ & R be a relation on A

$R = \{(a,b) \mid a - b = 5, \forall a, b \in A\}$

2. Universal relation: A relation R is said to be a universal relation if every element of set A is related to every other element of set A i.e., $R = A \times A$.

ex:- $A = \{1, 2, 3, 4, 5\}$ defined by

$R = \{(a,b) \mid |a-b| \geq 0\}$

3- Identity relation:- A relation R is said to be an identity relation if it contains only the ordered pairs where every element of set A is related to only itself. i.e. $R = \{(a,a)\}$

ex:- If $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3)\}$

4. Reflexive relation:

A relation R is reflexive if for all $x \in A$, $(x,x) \in R$

Let $A = \{1,2\}$ & $B = \{1,2,3\}$

$R = \{(1,1),(2,2),(1,3)\}$

5. Symmetric relation:-

A relation R is symmetric if $(a,b) \in R$ then $(b,a) \in R$

Ex:- $A = \{1,2\}$ & $B = \{1,2,3\}$

$R = \{(1,2),(2,1),(2,2)\}$

6. Transitive relation:-

A relation R is transitive if $(x,y) \in R$ & $(y,z) \in R$ then $(x,z) \in R$

ex:- $A = \{1,2\}$, $B = \{1,2,3\}$

$R = \{(1,2),(2,1),(1,1)\}$

7. Equivalence relation:-

A relation R is called equivalence relation in A, if

1] R is reflexive

2] R is symmetric

3] R is transitive.

8. Anti-symmetric relation:-

If aRb and bRa then $a=b$

eg:- Let R be a relation of $\subseteq$

$A \subseteq B$ ARB

$B \subseteq A$ BRA

$\Rightarrow A = B$.

Ex:-1

Let $A = \{2,3,4\}$ and $B = \{3,4,5,6,7\}$.
Assume a relation R from A to B such that $(x,y) \in R$ when x divides y. Determine R, its domain & range.

Sol.

$R = \{(2,4),(2,6),(3,3),(3,6),(4,4)\}$

Domain $= \{2,3,4\}$

Range $= \{3,4,6\}$

Ex 2:-

Let $A = \{1,2,3,4,5,6\}$. Define a relation R from A to A by $R = \{(x,y) \mid y = x+1\}$

i) Depict this relation using an arrow diagram

ii) Write domain, co-domain and range of R.

Sol:-

$y = x+1$

put $x = 1$, $y = 2$

$x = 2$, $y = 3$

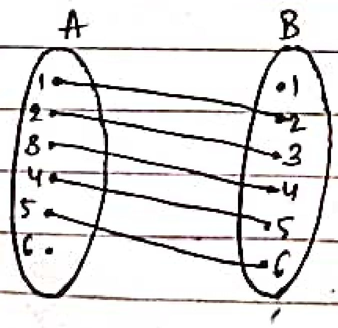
$x = 3$, $y = 4$

$x = 4$, $y = 5$

$x = 5$, $y = 6$

$R = \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$

i)



ii)

Domain $= \{1,2,3,4,5\}$

Range $= \{2,3,4,5,6\}$

Co-domain $= \{1,2,3,4,5,6\}$

Exercise problems :-

Determine whether each of the following relations are reflexive, symmetric and transitive.

(i) Relation R in the set $A = \{1, 2, 3, \ldots, 12, 13\}$ defined as $R = \{(x, y) : 3x - y = 0\}$

Given, $3x = y$

$y = 3x$

put $x = 1$, $y = 3$

$x = 2$, $y = 6$

$x = 3$, $y = 9$

$x = 4$, $y = 12$

$\vdots$

$x = 12$, $y = 36$

$x = 13$, $y = 39$

$R = \{(1,3), (2,6), (3,9), (4,12), \ldots, (12,36), (13,39)\}$

Neither reflexive nor symmetric nor transitive.

(ii) Relation R in the set N of natural numbers defined as $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$

Given, $y = x + 5$

put, $x = 1$, $y = 1 + 5 = 6$

$x = 2$, $y = 2 + 5 = 7$

$x = 3$, $y = 3 + 5 = 8$

$R = \{(1,6), (2,7), (3,8)\}$

Neither reflexive nor symmetric nor transitive

iii) Relation R in the set $A = \{1,2,3,4,5,6\}$ as
$R = \{(x,y) : y \text{ is divisible by } x\}$

$R = \{(1,1), (2,2), (2,1), (4,2), (3,3), (4,4), (5,5)$
$(6,6), (6,3), (6,2), (3,1), (4,1), (5,1), (6,1)\}$

R is reflexive
R is not symmetric i.e. $(2,1) \in R$ but $(1,2) \notin R$
R is transitive

Q:- Let N be the set of all natural numbers. R be a relation in N defined xRy iff $x+3y=12$.
Examine the relation R is (i) Reflexive (ii) Symmetric (iii) transitive

Sol:- Given, $R = \{(x,y) \mid x+3y=12 \ \forall \ x,y \in N\}$

$R = \{(9,1),(6,2),(3,3)\}$

$$x = 12-3y$$

put $y=1$, $x = 12-3 = 9$
$y=2$, $x = 12-6 = 6$
$y=3$, $x = 12-9 = 3$

R is not reflexive, because $(1,1), (2,2) \notin R$
R is not symmetric, because $(9,1) \in R \Rightarrow (1,9) \notin R$
R is not transitive.

Q:- Show that the relation $x \equiv y \pmod 5$ defined on the set of integers, I is an equivalence relation.

Sol:- [Congruence modulo m is a relation between two integers a and b, that states that m divides the difference between a and b

$$x \equiv y \pmod m \Rightarrow x - y = mk, \; k \in I]$$

* $x \equiv x \pmod 5$

$\Rightarrow (x - x) = 5k$

$\Rightarrow 0 = 5k$

$\Rightarrow k = 0 \in I$

$\therefore$ R is reflexive

* $x \equiv y \pmod 5$

$\Rightarrow (x - y) = 5k$

$\Rightarrow (y - x) = 5(-k), \; -k \in I$

$\therefore$ R is Symmetric

* $x \equiv y \pmod 5$

$\Rightarrow x - y = 5k_1 \rightarrow (1)$

$y \equiv z \pmod 5$

$\Rightarrow y - z = 5k_2 \rightarrow (2)$

(1)+(2)

$x - z = 5(k_1 + k_2)$

$x - z = 5k_3, \; k_3 \in I$

$x \equiv z \pmod 5$

$\therefore$ R is transitive.

Q:- Show that the relation R in the set A ~~of~~ $= \{1, 2, 3, 4, 5\}$ given by $R = \{(a,b) \mid |a-b| \text{ is even}\}$ is an equivalence relation.

Sol:- $R = \{(a,b) : |a-b| \text{ is even}\}$, where $a, b \in A$

* Since $|a-a| = |0| = 0$ and 0 is even

$\Rightarrow |a-a|$ is even

$\Rightarrow (a,a) \in R$

$\therefore$ R is reflexive

* Symmetric:-

WKT, $|a-b| = |b-a|$

if $|a-b|$ is even, then $|b-a|$ is also even

Hence, if $(a,b) \in R$ then $(b,a) \in R$

$\therefore$ R is symmetric

* If $|a-b|$ is even, then $(a-b)$ is even
if $|b-c|$ is even, then $(b-c)$ is even

Now, Sum of even numbers is also even

$a-b+b-c = (a-c)$ is even

Hence, $|a-c|$ is even

i.e. if $(a,b) \in R$ & $(b,c) \in R$,

then $(a,c) \in R$

$\therefore$ R is transitive.

Since R is reflexive, symmetric and transitive

$\therefore$ It is equivalence relation.

Q:- Give an example of a relation which is,

(i) Symmetric but neither reflexive nor transitive.

$A = \{1, 2, 3\}$

→ $R = \{(1,2),(2,1),(2,3),(3,2)\}$

(ii) Transitive but neither reflexive nor symmetric.

→ $R = \{(a,b) : a < b\}$

(iii) Reflexive and transitive but not symmetric.

→ $R = \{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$

(iv) Symmetric and transitive but not reflexive.

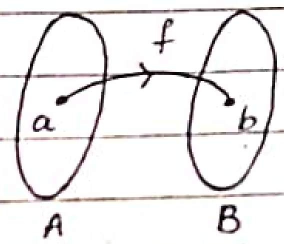
→ $R = \{(a,b) : a^3 > b^3\}$

Functions

Let A and B, be two non-empty sets. Then a relation f from a set A to a set B is called a function if every element in A has a unique image in B

We write $f: A \rightarrow B$, for each $a \in A$, there exists $b \in B$ such that $(a, b) \in f$.

The pictorial representation of f



If $(a, b) \in f$, we write $f(a) = b$.

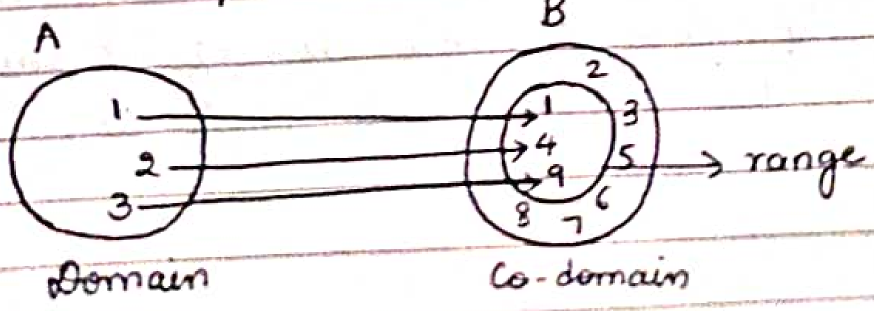
here, 'b' is called the image of a

'a' is called a pre image of b, under f.

Representation of Functions

- a set of ordered pairs
- an arrow diagram
- a table form
- a graphical form

$f(x) = x^2$



Types of Functions

1. One-One function (Injective function)

A function $f: X \to Y$ is defined to be one-one if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

2. Onto function (Surjective function)

A function $f: X \to Y$ is said to be onto, if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that

$$f(x) = y.$$

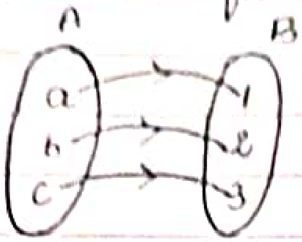
3. Bijective function

A function $f: X \to Y$ is said to be bijective if f is both one-one and onto.

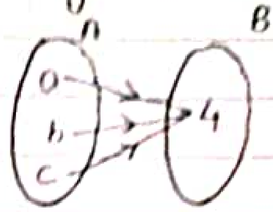
4. Many-one function

A function $f: X \to Y$ is said to be many-one, if two or more elements of the domain set X are connected to a single element in set Y

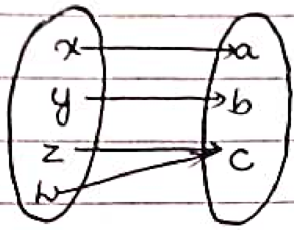
One to One function



Many to one



Onto function



Note:-

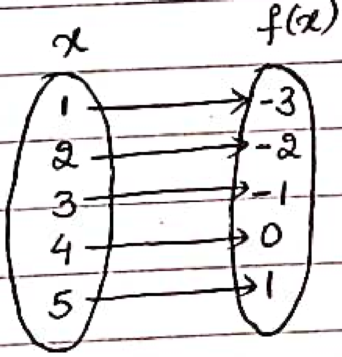
Number of functions

* One-One function : $^nP_m = \frac{n!}{(n-m)!}$ $n \geq m$

* Onto function : $\sum_{k=0}^{n} \{(-1)^k \cdot {}^nC_k \cdot (n-k)^m$, when $m \geq n$.

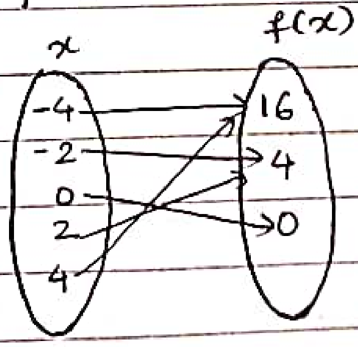
Examples:-

$f(x) = x-4$



One-One

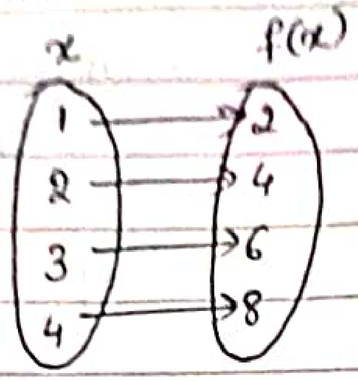
$f(x) = x^2$



(i) not one to one.

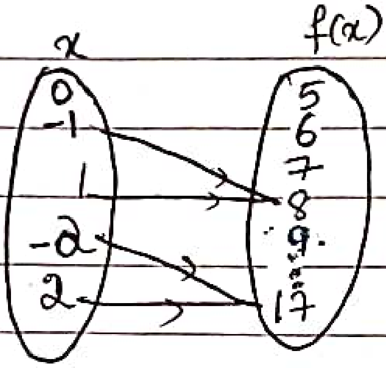
(ii) Onto.

$f(x) = 2x$



Onto-function.

$f(x) = 3x^2 + 5$



Many-One function.

Bijective function

↓

$f(x) = 2x + 1$

Equal functions :-

Consider two functions $f : A \to B$ and $g : A \to B$.
Then functions f and g will be equal functions if and only if their domain and range are the same.

i.e., for $f : A \to B$ and $g : A \to B$, $\forall\, a \in A$ there exists $b \in B$, such that $f(a) = g(a)$

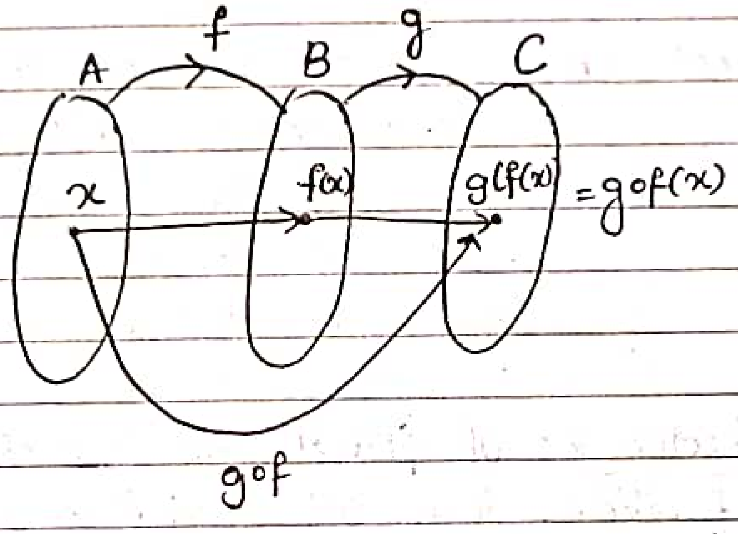
ex :- Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$

Composition of Functions

Let $f: A \to B$ and $g: B \to C$ be two functions then, a function $gof: A \to C$ is defined by

$$(gof)(x) = g(f(x)), \text{ for all } x \in A.$$

Here, $(gof)(x)$ is called the composition of f and g. [read as 'g of f of x']



ex:- If, $f(x) = x^2$ and $g(x) = x+3$. Then find $gof(x)$ and $fog(x)$

Sol:-

$$gof(x) = g(f(x))$$
$$= g(x^2)$$
$$gof(x) = x^2+3$$

$$fog(x) = f(g(x))$$
$$= f(x+3)$$
$$= (x+3)^2$$

Note:-

① $fog \neq gof$

② If f and g both are one-one function then fog is also one-one.

- If f and g both are onto function then fog is also onto.
- $(fog)^{-1} = g^{-1}of^{-1}$

Theorem 1 :- If $f: A \to B$ and $g: B \to C$ are one to one functions. Then, gof is also one to one function.

Proof :- Consider $(gof)(a_1) = (gof)(a_2)$, for any $a_1, a_2 \in A$

$$\Rightarrow g(f(a_1)) = g(f(a_2))$$

$$\Rightarrow f(a_1) = f(a_2) \quad [\because g \text{ is } 1-1]$$

$$\Rightarrow a_1 = a_2 \quad [\because f \text{ is } 1-1]$$

$\therefore gof$ is one-to one.

Theorem 2 :- If $f: A \to B$ and $g: B \to C$ are onto functions. Then, gof is also onto function.

Proof :-

Since $g: B \to C$ is onto
Suppose $c \in C$, then $\exists\ b \in B$
such that $g(b) = c$

Similarly, since $f: A \to B$ is onto
if $b \in B$, then $\exists\ a \in A$
such that $f(a) = b$

Now, $gof : A \to C$

$$gof = gof(a)$$
$$= g(f(a))$$
$$= g(b)$$

$gof = c$, So, for every $a \in A$, there is an image $c \in C$.

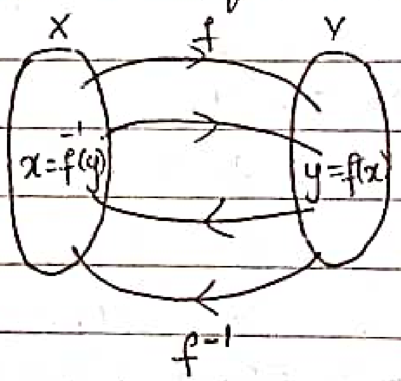
Thus, gof is onto.

Theorem 3:- If $f: A \to B$ and $g: B \to C$ are bijective functions. Then, $g \circ f$ is also bijective function.

Proof:- [Combine proof of Theorem 1 & Theorem 2]

Inverse or Invertible functions.

If $f: X \to Y$ is a bijection then there always exist pre-image $f^{-1}(y)$ of each element y of Y and this will be a unique element of Y



Ex:- If $f(x) = \frac{3x-5}{2x+5}$, find $f^{-1}(x)$

Steps to be followed

① Take $f(x)$ as y
② Interchange x & y
③ Solve for y
④ Replace y with $f^{-1}(x)$

Sol:- Let $y = f(x)$

① $\therefore \quad y = \frac{3x-5}{2x+5}$

$\cancel{(2x+5)y = 3x-5}$

$\cancel{2xy + 5y = 3x-5}$

$\cancel{2x}$

② Interchange x and y

$$x = \frac{3y-5}{2y+5}$$

(3) Solve for y

$$x(2y+5) = 3y-5$$
$$2xy + 5x = 3y-5$$
$$2xy - 3y = -(5x+5)$$
$$y(2x-3) = -(5x+5)$$
$$y = \frac{-(5x+5)}{2x-3}$$
$$\therefore f^{-1}(x) = -\frac{(5+5x)}{2x-3}.$$

Another method

Let $f^{-1}(x) = y$

$\Rightarrow x = f(y)$

$\Rightarrow x = \frac{3y-5}{2y+5}$

$\Rightarrow (2y+5)x = 3y-5$

$\Rightarrow 2xy + 5x = 3y-5$

$\Rightarrow 2xy - 3y = -(5x+5)$

$\Rightarrow y = \frac{-(5x+5)}{2x-3}$

$\Rightarrow f^{-1}(x) = \frac{-(5x+5)}{2x-3}$

Ex:- If $f: R \to R$, is defined by $f(x) = 5x-7$, Show that f is bijective and find the inverse functions.

Sol:- We have, $f(x) = 5x-7$

Let $f(x_1) = f(x_2) \Rightarrow 5x_1 - 7 = 5x_2 - 7$

$$\Rightarrow 5x_1 = 5x_2$$
$$\Rightarrow x_1 = x_2$$

Thus $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore f$ is one-one.

Let $y \in R$ and $f(x) = y$

$\Rightarrow y \in R$ and $5x - 7 = y$

$\Rightarrow y \in R$ and $5x = 7 + y$

$\Rightarrow y \in R$ and $x = \frac{y+7}{5} \in R$

Thus, for every $y \in R$, $\exists$ $x \in R$ such that

$f(x) = y$ i.e., $f\left(\frac{y+7}{5}\right) = y$.

$\therefore f$ is onto

Hence, f is both one-one and onto.

further, $f^{-1}(x) = y$

$\Rightarrow x = f(y)$

$\Rightarrow x = 5y - 7$

$\Rightarrow y = \frac{x+7}{5}$

$\therefore f^{-1}(x) = \frac{x+7}{5}$.

Ex:- Consider $f: R_+ \to [4,\infty)$ given by $f(x) = x^2+4$. Show that f is invertible and find inverse of f. Where R_+ is the set of all non-negative real numbers.

Sol:- A function is invertible iff it is both injective and surjective.

Suppose $f(x_1) = f(x_2)$

then, $x_1^2+4 = x_2^2+4$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is injective

Let y be any non-negative real number.
we need to show that $\exists\, x \in R_+ \ni f(x) = y$.

$f(x) = y = x^2+4$

Consider $x = \sqrt{y-4}$

Since $y \geq 4$, $y-4$ is non-negative and hence $\sqrt{y-4}$ is well-defined and non-negative. Thus $x \in R_+$.

$f(x) = [\sqrt{y-4}]^2 + 4 = y - 4 + 4 = y$

$\therefore f$ is surjective

$\therefore f$ is both injective & surjective.

$\therefore f$ is invertible

Inverse of f :-

$$y = x^2 + 4$$
$$x^2 = y - 4$$
$$x = \sqrt{y-4}$$

$$f^{-1}(y) = \sqrt{y-4}$$

Mathematical Induction.

Mathematical induction is a technique or method to prove mathematical statements or theorems which is thought to be true for each and every natural number n.

Principal of Mathematical induction.

Let $P(n)$ be a property that is defined for integers n, Any statement or property can be proved using the principal of Mathematical induction by following steps.

Step 1 :- Verify if the statement is true for trivial cases $(n=1)$ i.e. check if $p(1)$ is true.

Step 2 :- Assume that the statement is true for $n=k$ for some $k \geq 1$ i.e. $P(K)$ is true.

Step 3 :- If the truth of $P(k)$ implies the truth of $P(K+1)$, then the statement $P(n)$ is true for all $n \geq 1$

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Various steps used in Mathematical induction are named accordingly.

Assumption step

- Basis step : Prove $P(K)$ is true for $K=1$
- Inductive step (Let $P(K)$ is true for all k in N and $K>1$). Prove $P(K+1)$ is true using basic mathematical properties.

Ex:- For all $n \geq 1$, prove that, $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Sol:- Let given statement be $P(n)$,

$$P(n) : 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n=1$

$$P(1) : \frac{1(1+1)(2+1)}{6} = 1$$

Now, for any positive integer k, assume $P(K)$ to be true i.e.,

$$P(K) : 1^2+2^2+3^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

We shall now prove that $P(K+1)$ is also true,

So,

$$P(K+1) = P(K) + (K+1)^2$$

$$= \frac{K(K+1)(2K+1)}{6} + (K+1)^2$$

$$= (K+1)\left[\frac{2K^2+K+(K+1)}{6}\right]$$

$$= (K+1)\left[\frac{2K^2+K+6K+6}{6}\right]$$

$$= (K+1)\left(\frac{2K^2+7K+6}{6}\right)$$

$$= \frac{(K+1)(K+2)(2K+3)}{6}$$

$$= \frac{(k+1)\,((k+1)+1)\,(2(k+1)+1)}{6}$$

Thus P(k+1) is true, whenever P(k) is true for all natural numbers. Hence, by the process of mathematical induction, the given result is true for all natural numbers.

(2) Show by mathematical induction,

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}, \forall n \geq 1$$

Sol:- Let given statement be $p(n)$,

$$p(n): 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

For $n = 1$

$$P(1): 1^3 = \frac{1^2(1+1)^2}{4}$$

$$1 = \frac{2^2}{4} = \frac{4}{4}$$

$$1 = 1$$

LHS = RHS

$\therefore$ $p(n)$ is true for $n = 1$

Now, for any positive integer K, assume $p(K)$ to be true i.e;

$$p(K): 1^3 + 2^3 + 3^3 + \dots + K^3 = \frac{K^2(K+1)^2}{4}$$

We shall now prove that $p(K+1)$ is also true, so,

$$p(k+1): 1^3+2^3+3^3+\dots+(k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4}$$

$$\underbrace{1^3+2^3+3^3+\dots+k^3}+(k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$$\frac{k^2(k+1)^2}{4}+(k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$$\frac{(k+1)^2}{4}\left[k^2+4(k+1)\right] = \frac{(k+1)^2(k+2)^2}{4}$$

$$\cancel{k^2+4(k^2+1+2k) = k^2+2k+4}$$

$$\cancel{k^2+4k^2+4+8k}$$

$$\Rightarrow \quad k^2+4k+4 = k^2+4k+4$$

LHS = RHS

$\therefore$ $p(n)$ is true for $n=k+1$

$\therefore$ $p(n)$ is true for all $n \geq 1$

(3) Show that $1+3+5+\dots+(2n-1) = n^2$.

Step 1 :- $p(n): 1+3+5+\dots+(2n-1) = n^2$

for $n=1$,

$p(1): 1 = 1$

$\therefore$ $p(n)$ is true for $n=1$

Step 2 :- Assume that the result is true for $n=k$,

$p(k): 1+3+5+\dots+(2k-1) = k^2$

Step 3 :- Now check for $n=k+1$

$$p(k+1): \underbrace{1+3+5+\dots+(2k-1)}+(2(k+1)-1) = (k+1)^2$$

$$k^2+(2k+2-1) = (k+1)^2$$

$$k^2+2k+1 = (k+1)^2$$

$$(k+1)^2 = (k+1)^2$$

LHS = RHS $\therefore$ $p(n)$ is true for all

(4) For all $n \geq 1$, prove that, $1.2 + 2.3 + 3.4 + \ldots + n(n+1)$
$= \frac{n(n+1)(n+2)}{3}$

Sol:- $p(n): 1.2 + 2.3 + 3.4 + \ldots + n.(n+1) = \frac{n(n+1)(n+2)}{3}$

for $n=1$,

$p(1): \quad 1.2 = \frac{1(1+1)(1+2)}{3}$

$2 = \frac{1.2.\cancel{3}}{\cancel{3}}$

$2 = 2$

which is true.

Now, let's take a positive integer k, and assume p(K) to be true i.e.,

$p(K): 1.2 + 2.3 + 3.4 + \ldots + K(K+1) = \frac{K(K+1)(K+2)}{3}$

We shall now prove that p(K+1) is also true,

So,

$$P(K+1): 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + K(K+1) + (K+1)(K+2)$$

$$= \frac{K(K+1)(K+2)}{3} + (K+1)(K+2).$$

$$= \frac{K(K+1)(K+2) + 3(K+1)(K+2)}{3}$$

$$= \frac{(K+1)(K+2)[K+3)}{3}.$$

(5) For all integers $n \geq 0$, $2^{2n}-1$ is divisible by 3.

Sol:- Let $P(n)$: $2^{2n}-1$ is divisible by 3.

Base step:- For $n=0$

$$P(0) = 2^{2(0)}-1 = 2^0-1$$

$$= 1-1 = 0 \text{ is divisible by } 3$$

$\therefore$ $p(n)$ is true for $n=0$

Assumption step:- For any positive integer k

$p(k) = 2^{2k}-1$ is divisible by 3

[Definition of divisibility:- If m divides n, then $n = mq$, q is an integer]

$$\therefore\ 2^{2k}-1 = 3q\ ;\ q \in \mathbb{Z}$$

Induction step:- For $n=k+1$

$$p(k+1) = 2^{2(k+1)}-1$$

$$= 2^{2k+2}-1$$

$$= 2^{2k}\cdot 2^2 - 1$$

$$= 2^{2k}\cdot 4 - 1$$

$$= 2^{2k}(3+1)-1$$

$$= 2^{2k}\cdot 3 + 2^{2k}-1$$

$$= 2^{2k}\cdot 3 + 3q$$

$$= 3(2^{2k}+q),\quad 2^{2k}+q \in \mathbb{Z}$$

$\therefore$ $2^{2(k+1)}-1$ is divisible by 3.