Operations Research [OR]

UNIT - I

Introduction to OR & Linear programming

The operation research is used to describe the application of scientific method to the operation and management of organization involving people and other resources like money and machinery.

Definition of OR

Operations Research is the Systematic application of quantitative methods, techniques and tools to the analysis of problem involving the operations of systems.

Origin and development of OR.

- The term "Operations Research" was coined in 1940 by A.P. Rowe, a British Air Ministry Scientist.
- The phrase referred to the application of Scientific methods to analyse and improve the effectiveness of military operations.
- Operations Research is a '<u>war baby</u>'. It is because, the first problem attempted to solve in a systematic way was concerned with how to set the time fuse bomb to be dropped from an aircraft on to a Submarine.
- The main objective was to allocate limited resources in an effective manner to various military operations and to the activities within each operation.
- Due to the availability of faster and flexible computing facilities and the number of qualified O.R. professionals, it is now widely used in military.

business, industry, transportation, public health, crime investigation, etc.

(Significant features / Nature / Characteristics of OR) —

Different Phases (or) Methodology of OR.

The job of OR is to Examine, formulate, analyse, solve the problem of management in a scientific way.

The Methodology of OR follows a rigorous process that includes the following distinct phases:-

1. Formulating the problem: Identifying the objective, decision variables involved and the constraints that arise involving the decision variables.
2. Construction of a Mathematical model:
   Expressing the measure of effectiveness which may be total profit, total cost, utility etc., to be optimized by a mathematical function called objective function.
   Representing the constraints like budget constraints, raw materials constraints, resource constraints, quality constraints etc., by means of mathematical equations or inequalities.
3. Solving the Model constructed.
   Model is solved using various mathematical and statistical tools using the input data.

Controlling and Updating:-
A solution which is optimum today may not be so tomorrow. The values of the variables may change, new variables may emerge. The structural relationship between the variables may also undergo a change. All these are determined updating.

A solution from a model remains a solution only so long as the uncontrolled variables retain their values and the relationship between the variables does not change. Therefore controls must be established to indicate the limits within which the model and its solution can be considered as reliable. This is called Controlling.

4. Testing the model and its solution.

Validating the model :- Checking as far as possible either from the past available data or by expertise and experience data, whether the model gives a solution which can be used in practice.

5. Implementation:-

Implement using the solution to achieve the desired goal.

Significant features / Nature / Characteristics of OR :-

1. Systematic Approach :- OR uses a structured approach to solve problems, typically beginning with the formulation of the problem, constructing a model, Deriving solution from the model, Testing the Model, Establishing controls over the Solution, and implementation of the solution. This methodical approach ensures consistency and comprehensiveness in tackling complex issues.

2. Quantitative Approach :- The core of OR is based on quantitative techniques. It relies on mathematical models to formulate problems involving complex systems or processes.

3. Interdisciplinary Approach :- OR often involves teams of specialists from various disciplines, including mathematics, statistics, economics, engineering, and psychology. This interdisciplinary approach ensures that all aspects of a problem are considered and that solutions benefit from diverse perspectives.

4. Goal Oriented Approach :- OR is focused on achieving specific goals like minimizing costs, maximizing efficiency, or optimizing resource allocation. This goal-oriented approach ensures that the solutions developed are directly aligned with organizational or project objectives.

5. Scientific approach :- OR applies scientific methods to the management of organized systems in business, industry, government, and other enterprises.

6. Iterative Process :- O.R. often involves an iterative process of problem formulation, model development, solution derivation, testing, and implementation to refine and improve decisions-making.

7. Optimization :- O.R. aims to optimize or improve decision outcomes by finding the best possible Solution given constraints and objectives.

8. Modeling :- O.R. involves the development of mathematical models that represent real-world systems, processes, or problems to facilitate analysis & decision making.

9. <u>Broad Application</u> :-

OR is not confined to any specific type of problem or industry sector. It has broad applications across various fields such as business, engineering, health-care, transportation, logistics, finance, and government, showcasing its versatility.

10. <u>Improvement Focus</u> :- O.R. promotes a culture of continuous improvement by evaluating & refining decision making processes based on feedback and Performance metrics.

<u>Applications of OR</u>.

The Technique of O.R. have very wide application in various fields of business, industrial government and social sector. Few areas of application are mentioned below.

1. Marketing and sales.
   * Pricing optimization
   * Customer segmentation
   * Advertising Budget Allocation

2. Production Management.
   * Product mix and product proportioning.
   * Material handling facilities planning.
   * Design of information systems.
   * Quality control decisions.

3. Finance, Investments and Budgeting:

* Profit planning
* Cash flow analysis
* Investment decisions & risk analysis
* Dividend Policies
* Portfolio Analysis.

4. Defense:

* Optimum weaponry systems
* Optimum level of force deployment
* Transportation cost.
* Assignment suitability.

5. Research and Development:

* Selection criteria for specific project.
* Trade-off analysis for time-cost relationship and control of devllopment projects.

6. Personal management.

* Determination of optimum organisation level.
* Job evaluation & assignment analysis.
* Salary Criteria.
* Recruitment policies.

Tools of OR.

* Decision Tables
* Decision Trees
* Game theory
* Forecasting
* Queuing modes
* Linear Programming
* Dynamic programming
* Inventory models.
* Transportation models
* Simulation etc.

Limitations of OR.

* Time-consuming
* Lack of acceptance by decision makers.
* Assessments of uncertainties are difficult to obtain
* Cost - OR requires the use of specialized software and tools, which can be expensive.
* Incomplete information - OR requires complete & accurate information to make optimal decisions. However, in many cases, the information available is incomplete or inaccurate. This can lead to incorrect decisions.

General Solution Methods for Operations Research Problems:-

(i) Analytical Method:-
In this method, the classical mathematical tools for optimization, like calculus, finite differences are used for solving the O.R. problem. The solutions obtained are called analytical solutions to the problem.

(ii) Iterative method:- When some analytical methods are failed to give solution for O.R. problem, some numerical methods are used with iterative or trial and error procedure to solve the problem.

(iii) Monte-Carlo Method:- This is a simulation technique using statistical distributions functions & random numbers. Simulation models "imitate" the behaviour of the system over a period of time.

LINEAR PROGRAMMING PROBLEMS (LPP)

Linear Programming is a mathematical technique for optimum allocation of limited resources, such as labour, material, machine, money, energy and so on, to several competing activities such as products, services, jobs and so on, on the basis of a given criteria of optimality.

Structure of Linear Programming Model

- The general structure of the Linear Programming model essentially consists of three components
  1. The variables and their relationship - the variables are represented by $x_1, x_2, x_3, \ldots, x_n$ are called decision variables.
  2. The objective function of an LPP is a mathematical representation of the objective in terms a measurable quantity such as profit, cost, revenue, etc.

     Optimize (Maximize or Minimize) $Z = C_1X_1 + C_2X_2 + \ldots + C$

     where Z is the measure of performance variable, $X_1, X_2, \ldots X_n$ are the decision variables and $C_1, C_2, \ldots C_n$ are the parameters that give contribution to decision variables.
  3. The constraints :- These are the set of linear inequalities and/or equalities which impose restriction of the limited resources.

Mathematical Formulation of LPP.

Optimize (Maximize or Minimize) $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to constraints,

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \ (\le, =, \ge)\ b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \ (\le, =, \ge)\ b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \ (\le, =, \ge)\ b_m$

and $x_1, x_2, \dots, x_n \ge 0$

Guidelines for Formulating Linear Programming Model.

1. Identify and define the decision variable of the problem.
2. Define the objective function.
3. State the constraints to which the objective function should be optimized (i.e. Maximization or minimization)
4. Add the non-negative constraints (Here decision variables cannot be negative).

Problem 1 :

A manufacturer produces two types of models M_1 & M_2. Each model of type M_1 requires 4 hours of grinding and 2 hours of polishing ; whereas Each model of type M_2 requires 2 hours of grinding and 5 hours of polishing. The Manufacturer has 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. Profit on M_1 model is Rs. 3.00 and on model M_2 is Rs. 4.00. How should the manufacturer allocate his production capacity to the two types of models, so that he may make maximum profit in a week? Formulate it as LPP.

Solution:-

Machine	Model M_1	Model M_2	Availability
Grinding	4	2	$2 \times 40 = 80$
Polishing	2	5	$3 \times 60 = 180$
Profit	₹ 3	₹ 4	

Decision variables :- Let x_1 and x_2 be the number of units produced model M_1 & model M_2.

Objective function :-

Since profit on both the models is given, we have to maximize the profit.

$$\therefore \text{Max } Z = 3x_1 + 4x_2 \rightarrow (i)$$

Constraints :- There are two constraints one for grinding & other for polishing.

Two grinders are working.

$\therefore$ no. of hours available for grinding $= 2 \times 40 = 80$ hrs

model M_1 requires 4hrs of grinding &
model M_2 requires 2hrs of grinding.
Hence, the grinding constraint is given by,

$$4x_1 + 2x_2 \le 80 \rightarrow (ii)$$

Similarly, there are 3 polishers.
Total no. of hours available for polishing $= 3 \times 60$
$= 180$ hrs.

Model M_1 requires 2hrs of polishing,
whereas model M_2 requires 5hrs of polishing.

$$\therefore \quad 2x_1 + 5x_2 \le 180 \rightarrow (iii)$$

Non-negative restriction

$$x_1, x_2 \ge 0 \rightarrow (iv)$$

Finally we have,

$$\boxed{\begin{aligned} &\text{Max } Z = 3x_1 + 4x_2 \\ &\text{Subject to, } 4x_1 + 2x_2 \le 80 \\ &\qquad 2x_1 + 5x_2 \le 180 \\ &\qquad x_1, x_2 \ge 0 \end{aligned}}$$

Problem 2:-

Suppose an industry is manufacturing two types of products P1 and P2. The profits per kg of the two products are Rs.30 and Rs. 40 respectively. These two products require processing in three types of machines. The following table shows the available machine hours per day and the time required on each.

Profit/kg	P1 Rs.30	P2 Rs.40	Total available machine hours/day
Machine 1	3	2	600
Machine 2	3	5	800
Machine 3	5	6	1100

Solution:-

Decision Variables:- Let x_1 = amount of P1
x_2 = amount of P2

Objective function:-

In order to maximize profits, we establish the objectives function as,

$$\text{Max } Z = 30x_1 + 40x_2$$

where, 30 is the profit per kg of product P1.
40 is the profit per kg of product P2.

Constraints:-

- Machine 1 constraint:-

Machine 1 has a daily capacity of 600 hours, and products P_1 & P_2 require 3 and 2 hours respectively on this machine.

$$3x_1 + 2x_2 \leq 600$$

- Machine 2 constraint:-

Machine 2 has a daily capacity of 800 hours, & products P_1 & P_2 require 3 and 5 hours respectively on this machine.

$$3x_1 + 5x_2 \leq 800$$

- Machine 3 constraint:-

Machine 3 has a daily capacity of 1100 hours, & products P_1 & P_2 require 5 and 6 hours respectively on this machine.

$$5x_1 + 6x_2 \leq 1100$$

Non-negativity constraints:

The amount of products manufactured cannot be negative.

$$x_1 \geq 0, \; x_2 \geq 0$$

Thus, LPP model is as follows:

$$\boxed{\begin{aligned} &\text{Maximize}, \quad Z = 30x_1 + 40x_2 \\ &\text{Subject to}, \quad 3x_1 + 2x_2 \le 600 \\ &\qquad 3x_1 + 5x_2 \le 800 \\ &\qquad 5x_1 + 6x_2 \le 1100 \\ &\qquad x_1, x_2 \ge 0 \end{aligned}}$$

Problem 3

A company manufactures two products A & B. These products are processed in the same machine. It takes 10 mins to process 1 unit of product A & 2 mins for each unit of product B and the machine operates for a maximum 35 hours in a week.

Product A requires 1 kg and product B requires 0.5 kg of raw materials per unit, the supply of which is 600 kg per week. Market constraints on product B is known to be minimum of 800 units per week.

Product A cost Rs. 5 per unit and sold at Rs. 10 per unit, product B cost Rs. 6 per unit & can be sold in market at Rs. 8 per unit. Determine the number of unit of A & B per week to maximize the problem.

Solution:-

Decision variable :- Let x_1 and x_2 be the number of products A and B respectively.

Objective function :- Cost of product A per unit = Rs. 5

SP of product A = Rs. 10.

∴ Profit on one unit of product A = 10 − 5 = Rs. 5

Cost of product B per unit = Rs. 6

SP of product B = Rs. 8.

Profit on one unit of product B = 8 − 6

The Objective function is given by;

Maximize $Z = 5x_1 + 2x_2$.

Constraints.

To process product A it takes 10 min

To process product B it takes 2 min

Maximum hours is 35 hrs $= 35 \times 60 = 2100$ mins

$\therefore$ requirement constraint is,

$$10x_1 + 2x_2 \leq 2100$$

Raw material constraint

$$x_1 + 0.5x_2 \leq 600$$

Product B can be sold minimum of 800 kg.

$$x_2 \geq 800.$$

Finally the LPP is,

$$\boxed{\begin{aligned} &\text{Max } Z = 5x_1 + 2x_2 \\ &\text{Subject to, } 10x_1 + 2x_2 \leq 2100 \\ &\qquad x_1 + 0.5x_2 \leq 600 \\ &\qquad x_2 \geq 800 \\ &\qquad x_1, x_2 \geq 0 \end{aligned}}$$

4. The agricultural research institute suggested the farmer to spread out atleast 4800kg of special phosphate fertilizer & not less than 7200kg of a special nitrogen fertilizer to raise the productivity of crops in his fields. There are two sources for obtaining these - mixtures A & mixtures B. Both of these are available in bags weighing 100kg each & they cost Rs. 40 & Rs. 24 respectively. Mixture A contains phosphate & nitrogen equivalent of 20 kg & 80 kg respectively, while mixture B contains these ingredients equivalent of 50 kg each. Write this as an LPP & determine how many bags of each type the farmer should

buy in order to obtain the required fertilizer at minimum cost

Sol:-

i. Identify & define the decision variable of the problem:

Let x_1 & x_2 be the no. of bags of mixture A & mixture B.

ii. Define the Objective function.

The cost of mixture A & mixture B are given;

the objective function is to minimize the cost.

$$\text{Min } Z = 40x_1 + 24x_2$$

iii. State the constraints to which the objective function should be optimized.

Phosphate requirement, $20x_1 + 50x_1 \geq 4800$

Nitrogen requirement, $80x_1 + 50x_2 \geq 7200$

$$x_1, x_2 \geq 0$$

Finally, we have,

$$\boxed{\begin{array}{l} \text{min } Z = 40x_1 + 24x_2 \\ \text{ST constraints,} \\ 20x_1 + 50x_2 \geq 4800 \\ 80x_1 + 50x_2 \geq 7200 \\ x_1, x_2 \geq 0. \end{array}}$$

5. A person requires 10, 12 and 12 units chemicals A, B & C respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B & C respectively per jar. A dry product contains 1, 2 and 4 units of A, B & C per carton. If the liquid product sells for Rs. 3 per jar & the dry product sells for Rs. 2 per carton, how many of each should be purchased, in order to minimize the cost & meet the requirements?

Sol:-

i. Identify and define the decision variable of the problem

Let x_1 be the number of units of liquid products

& x_2 be the number of units of dry products.

ii. Define the objective function.

The cost of liquid and dry products are given;

The objective function is to minimize the cost.

$$Min.Z = 3x_1 + 2x_2$$

iii. State the constraints to which the objective function should be optimized. The above objective function is subjected to following three constraints.

$$5x_1 + 5x_2 \geq 10$$
$$2x_1 + 2x_2 \geq 12$$
$$x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Problem 6:-

A company manufactures two products P_1 and P_2, Profit per unit for P_1 is Rs.200 and for P_2 is Rs.300. Three raw materials M_1, M_2, M_3 are required. One unit of P_1 needs 5 units of M_1 and 10 units of M_2. One unit of P_2 needs 18 units of M_2 and 10 units of M_3. Availability is 50 units of M_1, 90 units of M_2 and 50 units of M_3. Formulate as LPP.

	P_1	P_2	Availability.
M_1	5		50
M_2	10	18	90
M_3		10	50
Profit	200	300	

Soln:- Maximize $Z = 200x_1 + 300x_2$

Subject to,

$5x_1 \le 50$

$10x_1 + 18x_2 \le 90$

$10x_2 \le 50$

$x_1, x_2 \ge 0$

7. Two vitamins A and B are to be given as health supplements on daily basis to students. There are two products Alpha & Beta which contain vitamins A and B. One unit of Alpha contains 2g of A and 1g of B. One unit of Beta contains 1g of A and 2g of B. Daily requirement for A and B are atleast 10g each. Cost per unit of Alpha is Rs.20 & Beta is Rs.30. Formulate LPP to satisfy the requirement at minimum cost.

Vitamins	Products		Min daily requirements	Cost per unit
	Alpha (g)	Beta (g)		
A	2	1	10	20
B	1	2	10	30

Decision variables:-

X_1 = No. of units of Alpha per day

X_2 = No. of units of Beta per day

Objective function:-

$\text{Min } Z = 20x_1 + 30x_2$

atleast - $\ge$

atmost - $\le$

Subject to constraints:-

Vit A constraint: $2x_1 + x_2 \ge 10$

Vit B constraint: $x_1 + 2x_2 \ge 10$

$x_1, x_2 \ge 0$.

8. Vitamins B1 and B2 are found in two foods F_1 & F_2. 1 unit of F1 contains 3 units of B1 and 4 units of B2. 1 unit of F_2 contains 5 units of B1 and 3 units of B2 respectively. The minimum daily prescribed consumption of B_1 & B_2 is 50 & 60 units respectively. The cost per unit of F_1 & F_2 is

Rs. 6 & Rs. 3 respectively. Formulate as LPP

Solution:-

Vitamins	F_1	F_2	
B_1	3	5	50
B_2	4	3	60
Cost	Rs. 6	Rs. 3	

Decision variables.

Let x_1 = no. of units of F_1 to consume

x_2 = no. of units of F_2 to consume.

Objective Function:-

- Minimize the cost of consumption, which can be expressed as:-

$$\min Z = 6x_1 + 3x_2.$$

Constraints:-

B_1 consumption constraint:-

The daily prescribed consumption of B_1 should be atleast 50 units.

$$3x_1 + 5x_2 \geq 50.$$

B_2 consumption constraint:-

The daily prescribed consumption of B_2 should be atleast 60 units.

$$4x_1 + 3x_2 \geq 60.$$

Non-negativity constraint:-

The no. of units of F_1 & F_2 consumed cannot be negative, so we have:-

$$x_1 \geq 0, \; x_2 \geq 0.$$

The formulated LPP can be summarized as follows, Minimize $Z = 6x_1 + 3x_2$

Subject to, $3x_1 + 5x_2 \geq 50$

$4x_1 + 3x_2 \geq 60$

$x_1 \geq 0, \; x_2 \geq 0$

GRAPHICAL METHOD OF SOLVING LPP.

Feasible Region:- The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of a LPP is called the feasible region for the problem.

Feasible solutions:- Points within and on the boundary of the feasible region represent feasible solutions of the constraints.

Infeasible Solution:- Any point outside the feasible region is called an infeasible solution.

Optimal solution:- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.

Working procedure for graphical Method.

Given an LPP,

Step 1:- Consider the inequality constraints as equalities. Draw the straight lines in the XOY plane corresponding to each equality and non-negativity restrictions.

Step 2:- Find the feasible region (or convex region) for the values of the variables which is the region bounded by the lines drawn in Step 1.

Step 3:- Find the points of intersection of the bounded lines by solving the equations of the corresponding lines.

Step 4:- Find the values of Z at all vertices of the permissible region.

Step 5:-

(i) For maximization problem, choose the vertex for which Z is maximum.

(ii) For minimization problem, choose the vertex for which Z is minimum.

1. Solve the following LPP by graphical Method.

Minimize, $Z = 20x_1 + 10x_2$

Subject to, $x_1 + 2x_2 \leq 40$

$3x_1 + x_2 \geq 30$

$4x_1 + 3x_2 \geq 60$

$x_1, x_2 \geq 0$.

Sol:- Replace all the inequalities of the constraints by equati

1. $x_1 + 2x_2 = 40$

If $x_1 = 0 \Rightarrow x_2 = 20$

If $x_2 = 0 \Rightarrow x_1 = 40$

$x_1 + 2x_2 = 40$ passes through $(0,20), (40,0)$

2. $3x_1 + x_2 = 30$

If $x_1 = 0 \Rightarrow x_2 = 30$

If $x_2 = 0 \Rightarrow x_1 = 10$

$3x_1 + x_2 = 30$ passes through $(0,30)$ $(10,0)$

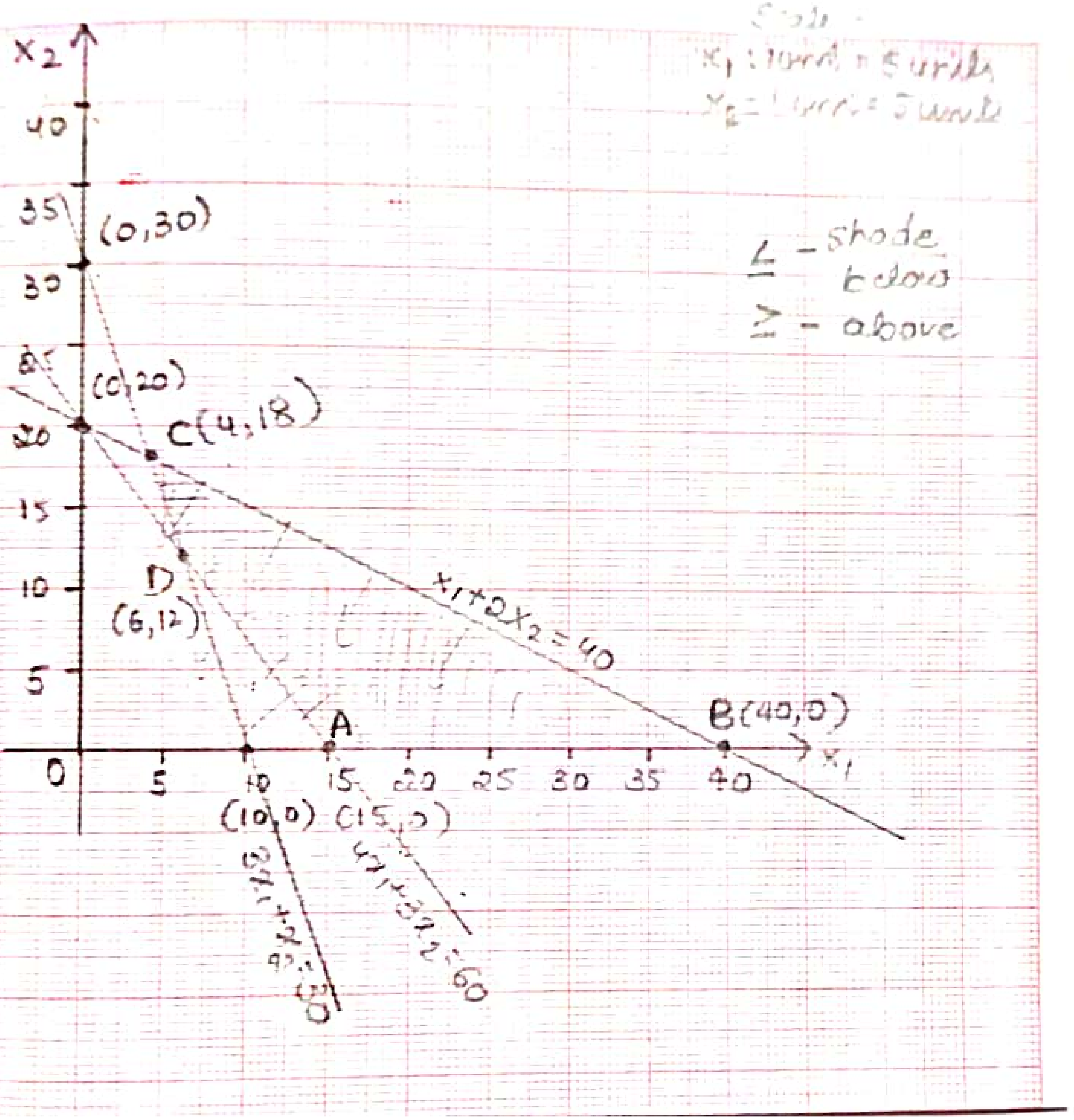
3. $4x_1 + 3x_2 = 60$

If $x_1 = 0 \Rightarrow x_2 = 20$

If $x_2 = 0 \Rightarrow x_1 = 15$

$4x_1 + 3x_2 = 60$ passes through $(0,20)$ $(15,0)$

Plot each eqn on the graph,



Scale:
x1: 1 cm = 5 units
x2 = 1 cm = 5 units
≤ - shade below
≥ - above
x2
x1
(0,30)
(0,20)
C(4,18)
D
(6,12)
A
B(40,0)
(10,0)
(15,0)
x1+2x2=40
3x1+x2=30
4x1+3x2=60

The feasible region is ABCD.

C & D are points of intersection of lines.

The value of the objective function at each of these extreme points is as follows:

Corner Points	Value of $Z = 20x_1 + 10x_2$
$A(15,0)$	$Z = 20 \times 15 + 10 \times 0 = 300$
$B(40,0)$	$Z = 20 \times 40 + 10 \times 0 = 800$
$C(4,18)$	$Z = 20 \times 4 + 10 \times 18 = 260$
$D(6,12)$	$Z = 20 \times 6 + 10 \times 12 = 240$ (Minimum Value)

∴ The minimum value of Z occurs at $D(6,12)$ & the optimal solution is $x_1 = 6$, $x_2 = 12$

$$(x_1 + 2x_2 = 40) \times 3 \qquad 3x_1 + 6x_2 = 120$$
$$(3x_1 + x_2 = 30) \times 1 \qquad 3x_1 + x_2 = 30$$
$$(-)\ (-)\ (-)$$
$$5x_2 = 90$$
$$x_2 = 18.$$

$$x_1 + 2(18) = 40$$
$$x_1 = 40 - 36$$
$$x_1 = 4$$

2. Find the maximum value of $Z = 5x_1 + 7x_2$

Subject to the constraints,

$x_1 + x_2 \leq 4$

$3x_1 + 8x_2 \leq 24$

$10x_1 + 7x_2 \leq 35$

$x_1, x_2 \geq 0$

Sol:- Replace all the inequalities of the constraints forming equations.

1. $x_1 + x_2 = 4$

If $x_1 = 0 \Rightarrow x_2 = 4$

If $x_2 = 0 \Rightarrow x_1 = 4$

$x_1 + x_2 = 4$ passes through $(0,4), (4,0)$

2. $3x_1 + 8x_2 = 24$

If $x_1 = 0 \Rightarrow x_2 = 3$

If $x_2 = 0 \Rightarrow x_1 = 8$

$3x_1 + 8x_2 = 24$ passes through $(0,3), (8,0)$

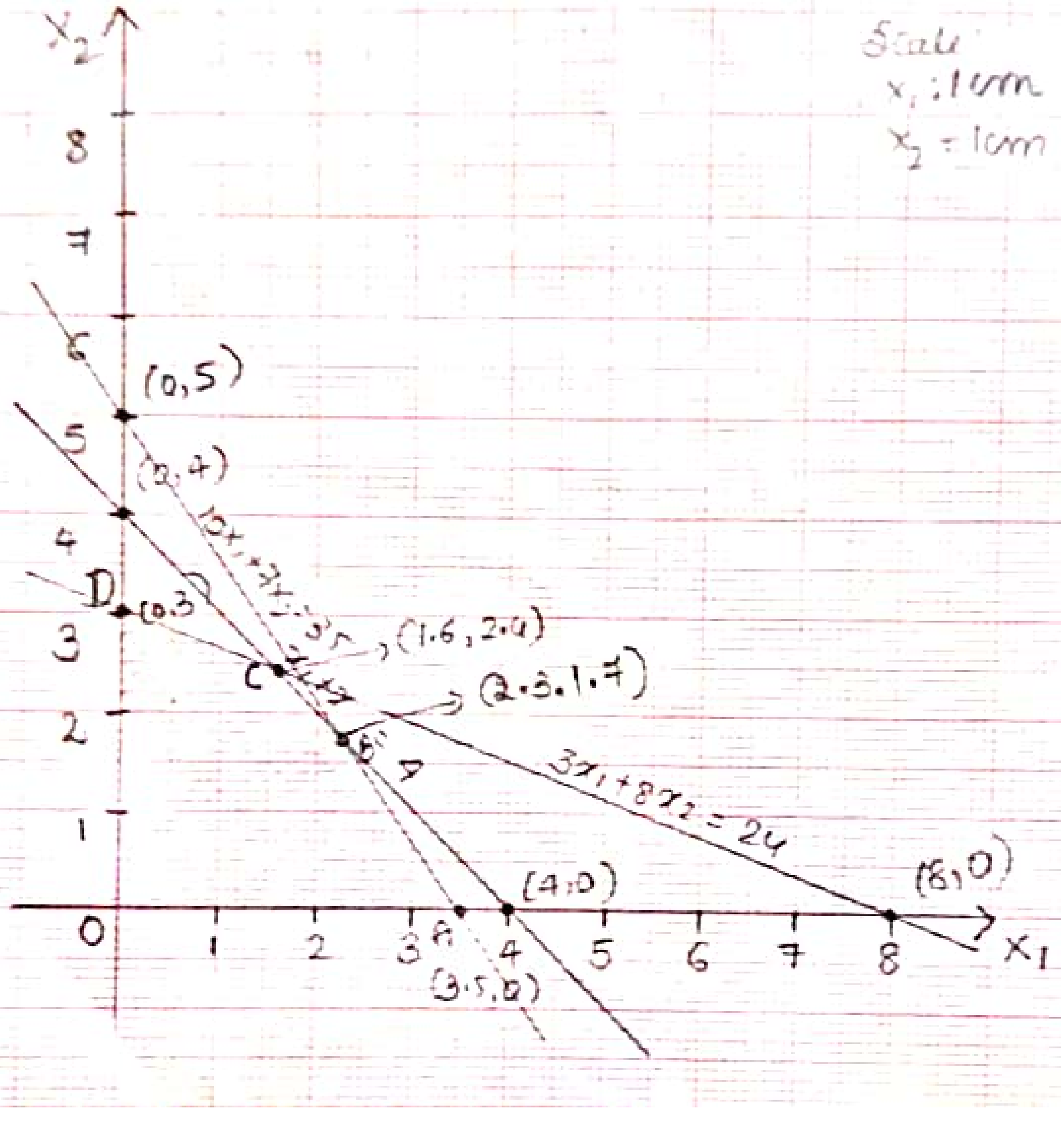
3. $10x_1 + 7x_2 = 35$

If $x_1 = 0 \Rightarrow x_2 = 5$

If $x_2 = 0 \Rightarrow x_1 = 3.5$

$10x_1 + 7x_2 = 35$ passes through $(0,5)(3.5,0)$

Scale
x_1 : 1cm = 1unit
x_2 = 1cm = 1unit



The Value of the objective function at each of these extreme points is as follows.

Corner Points	Objective function. $Z = 5x_1 + 7x_2$
$O(0,0)$	$5(0)+7(0)=0$
$A(3.5,0)$	$5(3.5)+7(0)=17.5$
$B(2.3,1.7)$	$5(2.3)+7(1.7)=23.4$
$C(1.6,2.4)$	$5(1,6)+7(2.4)=24.8$
$D(0,3)$	$5(0)+7(3)=21$

The maximum value of the objective function $Z=24.8$ occurs at the extreme point $(1.6, 2.4)$

Hence, the optimal solution to the given LP problem is :-

$x_1 = 1.6$

$x_2 = 2.4$

$\max Z = 24.8$.

3. Minimize $2000x_1 + 1500x_2$

Subject to :

$6x_1 + 2x_2 \geq 8$

$2x_1 + 4x_2 \geq 12$

$4x_1 + 12x_2 \geq 24$

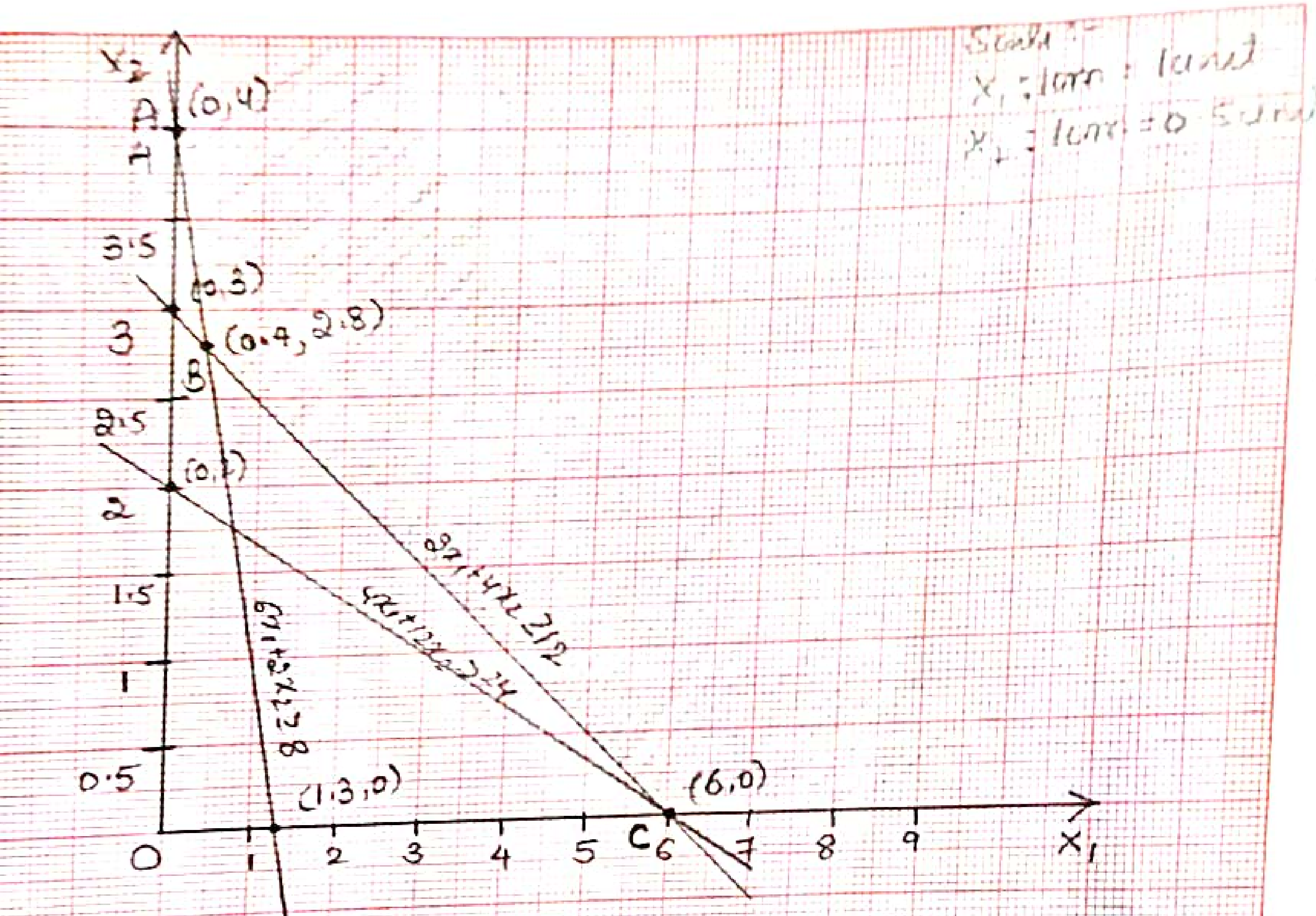
$x_1 \geq 0, \; x_2 \geq 0$

Sol:- Replace all the inequalities of the constraints by forming equations.

$6x_1 + 2x_2 = 8$, passes through the points $(1.3,0)$, $(0,4)$

$2x_1 + 4x_2 = 12$, passes through the points $(6,0)$, $(0,3)$

$4x_1 + 12x_2 = 24$, passes through the points $(6,0)$, $(0,2)$


Scale :-
X1 : 1cm = 1unit
X2 : 1cm = 0.5 unit
X2
A (0,4)
(0,3)
B (0.4, 2.8)
(0,2)
3x1+4x2 ≥ 12
4x1+12x2 ≥ 24
6x1+2x2 ≥ 8
(1.3,0)
C (6,0)
O
X1
0.5
1
1.5
2
2.5
3
3.5
4
1
2
3
4
5
6
7
8
9

The value of the objective function at each of these extreme points is as follows :-

Corner points	Objective function, $Z = 2000x_1 + 1500x_2$
A (0,4)	$2000(0) + 1500(4) = 6000$
B (0.4, 2.8)	$2000(0.4) + 1500(2.8) = 5000$ minimum
C (6,0)	$2000(6) + 1500(0) = 12000$

Problem has an unbounded solution.

4. Solve the following LPP graphically & interpret the result,

$$\text{Max } Z = 8x_1 + 16x_2$$

Subject to : $x_1 + x_2 \leq 200$

$$x_2 \leq 125$$

$$3x_1 + 6x_2 \leq 900$$

$$x_1, x_2 \geq 0$$

Sol:- Replace all inequalities by equations.

1. $x_1 + x_2 = 200$

If $x_1 = 0$, $x_2 = 200$

If $x_2 = 0$, $x_1 = 200$

$x_1 + x_2 = 200$ passes through $(0,200)$, $(200,0)$

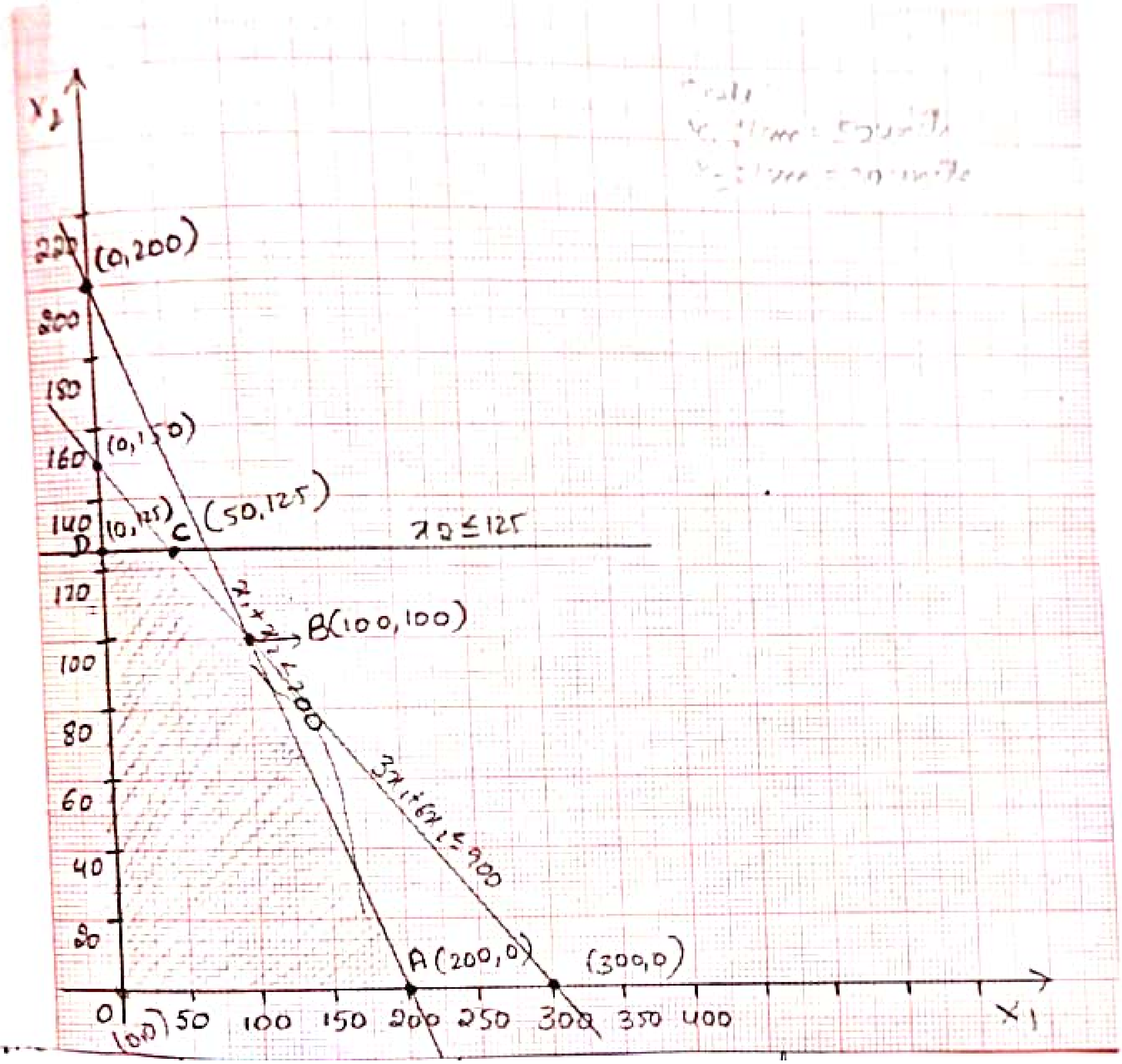
2. $x_2 = 125$

3. $3x_1 + 6x_2 = 900$

If $x_1 = 0 \Rightarrow x_2 = 150$

If $x_2 = 0 \Rightarrow x_1 = 300$

$3x_1 + 6x_2 = 900$ passes through $(0,150), (300,0)$



The value of the objective function at each of these extreme points follows.

Corner points	$Z = 8x_1 + 16x_2$
O (0,0)	$8(0)+16(0)=0$
A (200,0)	$8(200)+16(0)=1600$
B (100,100)	$8(100)+16(100)=2400$
C (50,125)	$8(50)+16(125)=2400$
D (0,125)	$8(0)+16(125)=2000$

The maximum value of the objective function $Z=2400$ occurs at 2 extreme points.
Hence, problem has multiple optimal solutions & max $Z=2400$

5. Find the solution using graphical method

Max $Z = 15x_1 + 10x_2$

Subject to,

$4x_1 + 6x_2 \le 360$

$3x_1 \le 180$

$5x_2 \le 200$

$x_1, x_2 \ge 0$.

Soln:- Replace all inequalities by eqns.

1. $4x_1 + 6x_2 = 360$

If $x_1 = 0 \Rightarrow x_2 = 60$

If $x_2 = 0 \Rightarrow x_1 = 90$

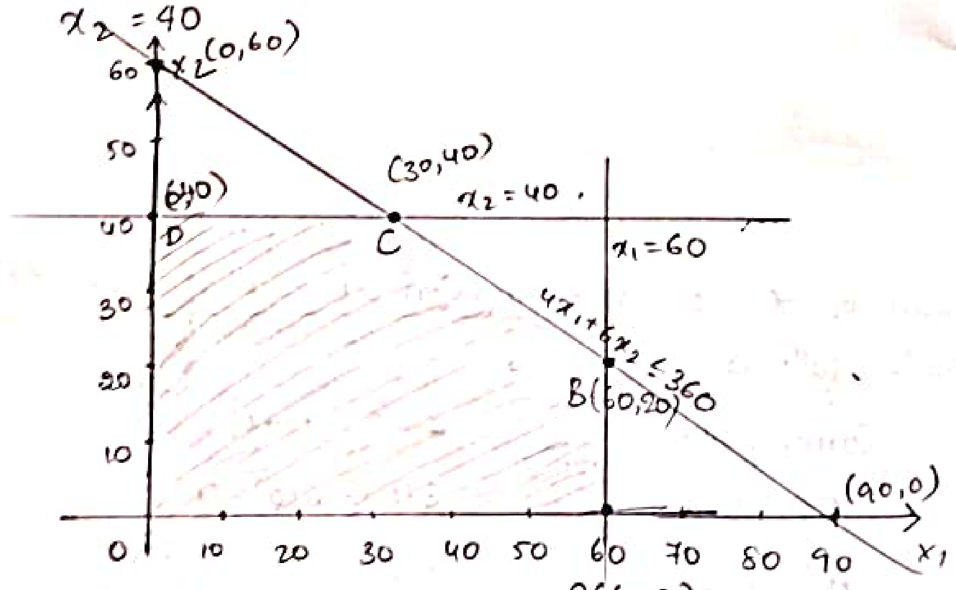
$4x_1 + 6x_2 = 360$ passes through $(0,60)$ $(90,0)$

2. $3x_1 = 180$

$x_1 = 60$

3. $5x_2 = 200$

$x_2 = 40$



The value of the objective function at each of these extreme points is as follows,

Corner points	$Z = 15x_1 + 10x_2$
$O(0,0)$	$15(0) + 10(0) = 0$
$A(60,0)$	$15(60) + 10(0) = 900$

$B(60,20)$ $\qquad 15(60)+10(20) = \underline{1100}$.

$C(30,40)$ $\qquad 15(30)+10(40) = 850$

$D(0,40)$ $\qquad 15(0)+10(40) = 400$

The maximum value of the Objective function $z = 1100$ occurs at the extreme point $(60,20)$.

Hence, the optimal solution to the given LP problem is,

$x_1 = 60,\ x_2 = 20$ & max $z = 1100$.

6. Solve the following LPP by graphical method

Maximize $Z = 10x_1 + 8x_2$

Subject to constraints $2x_1 + x_2 \leq 20$

$x_1 + 3x_2 \leq 30$

$x_1 - 2x_2 \geq -15$

$x_1, x_2 \geq 0$

Sol:- 1. To draw constraint $2x_1 + x_2 \leq 20 \rightarrow (1)$

treat it as $2x_1 + x_2 = 20$

when $x_1 = 0$ then $x_2 = 20$

when $x_2 = 0$ then $x_1 = 10$

$\therefore$ $2x_1 + x_2 \leq 20$ passes through $(0,20), (10,0)$

2. To draw constraint $x_1 + 3x_2 \leq 30 \rightarrow (2)$

Treat it as $x_1 + 3x_2 = 30$

when $x_1 = 0$ then $x_2 = 10$

when $x_2 = 0$ then $x_1 = 30$

$\therefore$ $x_1 + 3x_2 \leq 30$ passes through $(0,10), (30,0)$

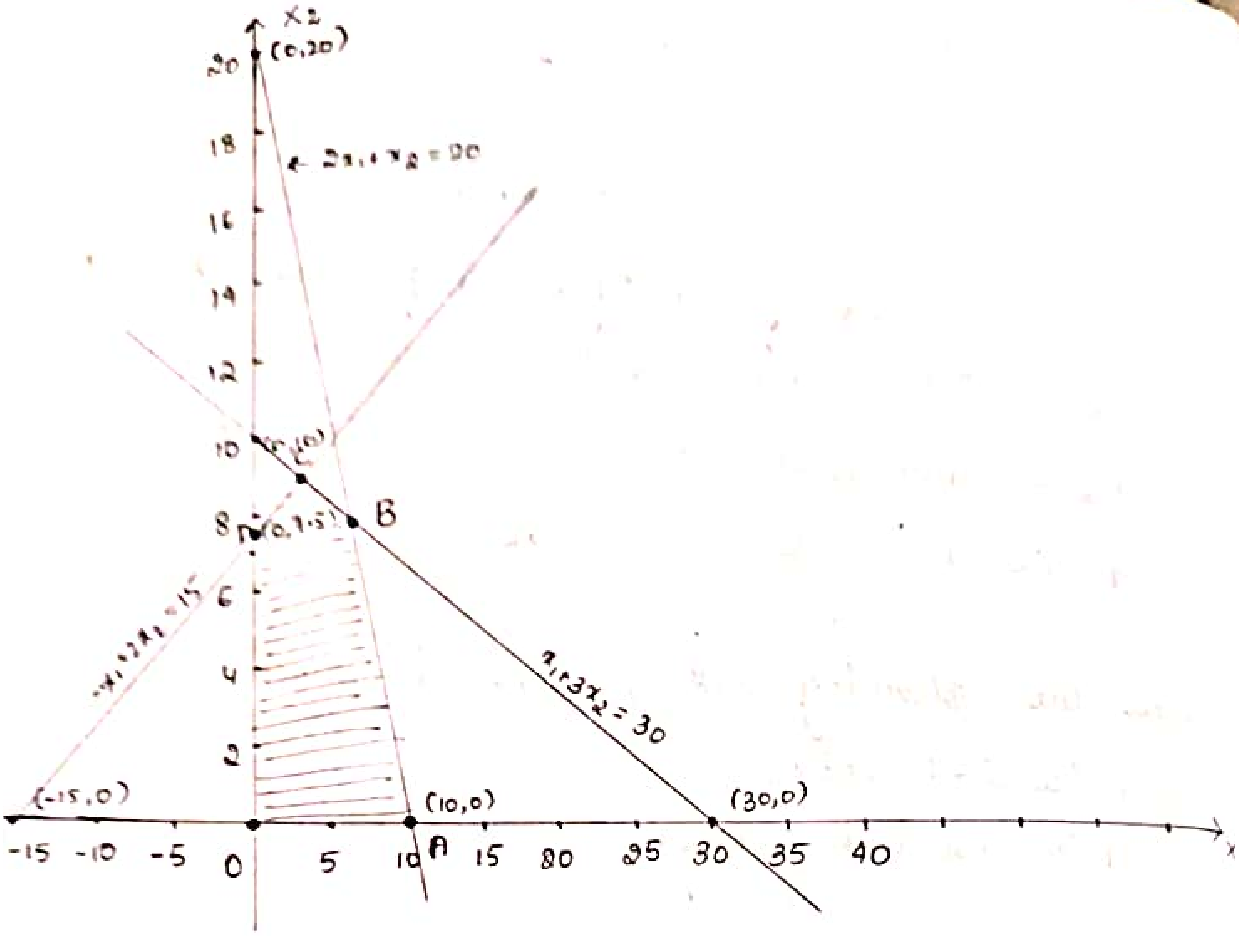
3. To draw constraint $x_1 - 2x_2 \geq -15$

treat it as $x_1 - 2x_2 = -15$

when $x_1 = 0$ then $x_2 = 7.5$

when $x_2 = 0$ then $x_1 = -15$

$\therefore$ $x_1 - 2x_2 \geq -15$ passes through $(0,7.5)$ $(-15,0)$



The value of the objective function at each of these extreme points is as follows :

Corner points	Objective function $z=10x_1+8x_2$
O (0,0)	$10(0)+8(0)=0$
A (10,0)	$10(10)+8(0)=100$
B (6,8)	$10(6)+8(8)=124$
C (3,9)	$10(3)+8(9)=102$
D (0,7.5)	$10(0)+8(7.5)=60$

The maximum value of the objective function $z=124$ occurs at the extreme point (6,8).

Hence, the optimal solution to the given LP problem is : $x_1=6$, $x_2=8$ & max $z=124$.

Simplex Method

If the LPP has larger number of variables, the suitable method for solving is simplex method.

The simplex method is an iterative process, through which it reaches ultimately to the minimum value or maximum value of the objective function.

Simplex method Algorithm

Assuming the existence of an initial basic feasible solution an optimal solution to any LPP by simplex method is found as follows.

Step 1:- Check whether the objective function is to be maximized or minimized. If it is to be minimized, then convert it into a problem of maximization.

by Minimize $Z = -$ Maximize Z.

Step 2:- Check whether all b_i's are positive. If any of the b_i's are negative, multiply both sides of that constraint by -1 so as to make its right hand side positive.

Step 3:- By introducing slack/surplus variables, convert the inequality constraints into equations & express the given LPP into its standard form.

Step 4:- Find an initial basic feasible solution and express the above information conveniently in the following simplex method.

		C_j	(C_1	C_2	C_3 ...			)
C_B	B	X_B	x_1	x_2	x_3 ...	S_1	S_2	S_3
C_{B1}	S_1	b_1	a_{11}	a_{12}	a_{13} ...	1	0	0
C_{B2}	S_2	b_2	a_{21}	a_{22}	a_{23} ...	0	1	0
C_{B3}	S_3	b_3	a_{31}	a_{32}	a_{33} ...	0	0	1
⋮		⋮						
$Z_j - C_j$		Z_0	$Z_1 - C_1$	$Z_2 - C_2$				

where C_j - now denotes the co-efficients of the variables in the objective function.

C_B - column denotes the co-efficients of the basic variables in the objective function.

B - column denotes the basic variables.

X_B - column denotes the values of the basic variables

The row $(Z_j - C_j)$ denotes the net evaluations index for each column.

Step 5:- Compute the net evaluations by using the relation $Z_j - C_j = C_B X_j - C_j$

i. If all $(Z_j - C_j) \geq 0$, then the current basic feasible solution X_B is optimal.

ii. If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal, go to the next step.

Step 6:- To find the entering variable.

The entering variable is the non-basic variable corresponding to the most negative value of $(Z_j - C_j)$

The entering variable column is known as the key column or pivot column which is shown marked with an arrow at the bottom. If more than one variable has the same most negative $(Z_j - C_j)$, any of these variables may be selected arbitarily as the entering variable.

Step 7:- To find the leaving variable.

Compute the ratio, $\theta = \min\left(\frac{X_B}{x_i}, x_i > 0\right)$

The leaving variable row called the key row or pivot row or pivot equation

The element at the intersection of the pivot column & pivot row is called the pivot element or key element

Steps:- Drop the leaving variable and introduce the entering variable along with its associated value under C_B column convert the pivot element to unity by dividing the pivot eqn by the pivot element. and all the other elements in its column to zero by making use of

New pivot equation = old pivot equation ÷ pivot element.

New eqn = Old eqn − (corresponding column co-efficient) × (New pivot equation)

Step 9:-

Go to step (5) and repeat the procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

Q1. Use Simplex method to solve LPP.

$\leq$ - introduce slack variable.

$\geq$ - surplus variable.

Maximize $Z = 4x_1 + 10x_2$

Subject to: $2x_1 + x_2 \leq 50$

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90$

$x_1, x_2 \geq 0.$

Sol:- Write Standard form,

Max $Z = 4x_1 + 10x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to: $2x_1 + x_2 + S_1 + 0S_2 + 0S_3 = 50$

$2x_1 + 5x_2 + 0S_1 + S_2 + 0S_3 = 100$

$2x_1 + 3x_2 + 0S_1 + 0S_2 + S_3 = 90$

and $x_1, x_2, S_1, S_2, S_3 \geq 0$

Since there are 3 equations with 5 variables, the initial basic feasible solution is obtained by equating $(5-3) = 2$ variables to zero.

$\therefore$ The initial basic feasible solution is $S_1 = 50, S_2 = 100, S_3 = 90$ ($x_1 = 0, x_2 = 0$, non basic)

The initial simplex table is given by,

entering variable

Initial:-

		C_j	4	10	0	0	0	Min ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	$\theta = \frac{X_B}{x_2}, x_2 > 0$
0	S_1	50	2	1	1	0	0	50
0	S_2	100	2	(5)	0	1	0	20 ←
0	S_3	90	2	3	0	0	1	30
$Z_j - C_j$		0	-4	-10 ↑	0	0	0	

Departing variable

$= \Sigma C_B X_j - C_j$

pivot (or) key column

Entering variable $= x_2$

Departing variable $= S_2$

Key (or) pivot element $= 5$.

$R_2(\text{new}) = R_2(\text{old}) \div 5$

R_2(old)	100	2	5	0	1	0
R_2(new)	20	2/5	1	0	1/5	0

$R_1(\text{new}) = R_1(\text{old}) - R_2(\text{new})$

R_1(old)	50	2	1	1	0	0
R_2(new)	20	2/5	1	0	1/5	0
R_1(new)	30	8/5	0	1	-1/5	0

$R_3(\text{new}) = R_3(\text{old}) - 3R_2(\text{new})$

R_3(old)	90	2	3	0	0	1
R_2(new)	20	2/5	1	0	1/5	0
$3R_2$(new)	60	6/5	3	0	3/5	0
R_3(new)	30	4/5	0	0	-3/5	1

First Iteration:

		C_j	4	10	0	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	30	8/5	0	1	-1/5	0
10	x_2	20	2/5	1	0	1/5	0
0	S_3	30	4/5	0	0	-3/5	1
Z_j-C_j		200	0	0	0	2	0

Since all $(Z_j - C_j)$'s ≥ 0 the current basic feasible solution is optimal.

$\therefore$ The Optimal Solution is Max $Z = 200$, $x_1 = 0$, $x_2 = 20$.

Q2. Find the non-negative values of x_1, x_2 and x_3 which maximize $Z = 3x_1 + 2x_2 + 5x_3$

Subject to, $x_1 + 4x_2 \leq 420$

$3x_1 + 2x_3 \leq 460$

$x_1 + 2x_2 + x_3 \leq 430$

$x_1, x_2, x_3 \geq 0$.

Sol:- Standard form of the LPP,

$\text{Max } Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to, $x_1 + 4x_2 + 0x_3 + S_1 + 0S_2 + 0S_3 = 420$

$3x_1 + 0x_2 + 2x_3 + 0S_1 + S_2 + 0S_3 = 460$

$x_1 + 2x_2 + x_3 + 0S_1 + 0S_2 + S_3 = 430$

and $x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$.

Since there are 3 equations with 6 variables, the initial basic feasible solution is obtained by eq^n (6-3)-3 variables to zero.

∴ The initial basic feasible solution is $S_1 = 420, S_2 = 460, S_3 = 430$ ($x_1 = x_2 = x_3 = 0$, non basic)

The initial simplex table is given by, . .

		C_j	3	2	5	0	0	0	Min ratio
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	$\theta = \frac{X_B}{x_3}, x_3 > 0$
0	S_1	420	1	4	0	1	0	0	—
0	S_2	460	3	0	2	0	1	0	230 ←
0	S_3	430	1	2	1	0	0	1	430
$Z_j - C_j$		0	-3	-2	-5 ↑	0	0	0	

Entering variable = x_3

Departing variable = S_2

Pivot / key element = 2

* $R_2(\text{new}) = R_2(\text{old}) \div 2$

R_2(old)	460	3	0	2	0	1	0
R_2(new)	230	3/2	0	1	0	1/2	0

* $R_1(\text{new}) = R_1(\text{old})$
* $R_3(\text{new}) = R_3(\text{old}) - R_2(\text{new})$

R_3(old)	430	1	2	1	0	0	1
R_2(new)	230	3/2	0	1	0	1/2	0
R_3(new)	200	-1/2	2	0	0	-1/2	1

First iteration :-

		C_j	3	2	5	0	0	0	Min ratio
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_i
0	S_1	420	1	4	0	1	0	0	105
5	x_3	230	1.5	0	1	0	0.5	0	-
0	S_3	200	-0.5	2	0	0	-0.5	1	100 ←
	Z_j-C_j	1150	4.5	-2 ↑	0	0	2.5	0	

Entering variable = x_2
Departing variable = S_3
Pivot element = 2.

* $R_3(new) = R_3(old) \div 2$

$R_3(old)$	200	-0.5	2	0	0	-0.5	1
$R_3(new)$	100	-0.25	1	0	0	-0.25	0.5

* $R_1(new) = R_1(old) - 4R_3(new)$

$R_1(old)$	420	1	4	0	1	0	0
$4R_3(new)$	400	-1	4	0	0	-1	2
* $R_1(new)$ =	20	2	0	0	1	1	-2

$R_2(new) = R_2(old)$

Second iteration :-

		C_j	3	2	5	0	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
0	S_1	20	2	0	0	1	1	-2
5	x_3	230	1.5	0	1	0	0.5	0
2	x_2	100	-0.25	1	0	0	-0.25	0.5
	Z_j-C_j	1350	4	0	0	0	2	1

Since all $(Z_j-C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The Optimal solution is,

$$Max\ Z = 1350,\ x_1 = 0,\ x_2 = 100,\ x_3 = 230.$$

Q3. Solve the following LPP by simplex method.

Minimize $= 8x_1 - 2x_2$

Subject to, $-4x_1 + 2x_2 \le 1$

$5x_1 - 4x_2 \le 3$

and $x_1, x_2 \ge 0$

Sol: $Max\ Z = -8x_1 + 2x_2$

Subject to, $-4x_1 + 2x_2 \le 1$

$5x_1 - 4x_2 \le 3$

& $x_1, x_2 \ge 0$

The problem is converted to standard form by adding slack variables.

After introducing slack variables,

$Max\ Z = -8x_1 + 2x_2 + 0S_1 + 0S_2$

Subject to,

$-4x_1 + 2x_2 + S_1 = 1.$

$5x_1 - 4x_2 + S_2 = 3$

and $x_1, x_2, S_1, S_2 \ge 0$

2 eq^ns
4 variables
$(4-2) = 2$
$x_1 = x_2 = 0.$

Initial Simplex table,

			c_1	c_2	c_3	c_4	
		C_j	-8	2	0	0	Min ratio $\theta = x_B/x_2,\ x_2 > 0$
C_B	B	x_B	x_1	x_2	S_1	S_2	
0	S_1	1	-4	(2)	1	0	$1/2 = 0.5$ ←
0	S_2	3	5	-4	0	1	—
$Z_j - C_j$		0	8	-2 ↑	0	0	

$Z_j = \Sigma C_B x_j$

$Z_1 = \Sigma C_B x_1$

$Z_1 = (0+0)$

Entering variable $= x_2$

Departing variable $= S_1$

Key / pivot element $= 2$

$R_1(\text{new}) = R_1(\text{old}) \div 2$

$R_1(\text{old})$	1	-4	2	1	0
$R_1(\text{new})$	0.5	-2	1	0.5	0

$R_2(\text{new}) = R_2(\text{old}) + 4\,R_1(\text{new})$

$R_2(\text{old})$	3	5	-4	0	1
$R_1(\text{new})$	0.5	-2	1	0.5	0
$4R_1(\text{new})$	2	-8	4	2	0
$R_2(\text{new})$	5	-3	0	2	1

First iteration :-

		C_j	-8	+2	0	0
C_B	B	x_B	x_1	x_2	s_1	s_2
2	x_2	0.5	-2	1	0.5	0
0	s_2	5	-3	0	2	1
$Z_j - C_j$		1	4	0	1	0

Since all $Z_j - C_j \geq 0$

Hence, optimal solution is arrived with value of variable as :-

$x_1 = 0,\ x_2 = 0.5$

$\text{Max } Z = 1$

$\therefore \text{Min } Z = -1$

Q9 Solve the LPP by simplex method.

Maximize $Z = 2x_1 + 2x_2 + 4x_3$

Subject to constraints,

$2x_1 + 3x_2 + x_3 \le 240$

$x_1 + 3x_2 + x_3 \le 300$

$x_1 + x_2 + 3x_3 \le 300$

$x_1, x_2, x_3 \ge 0$

Sol- Standard form,

$\text{Max } Z = 2x_1 + 2x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to,

$2x_1 + 3x_2 + x_3 + S_1 + 0S_2 + 0S_3 = 240$

$x_1 + 3x_2 + x_3 + 0S_1 + S_2 + 0S_3 = 300$

$x_1 + x_2 + 3x_3 + 0S_1 + 0S_2 + S_3 = 300$

$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0.$

Since there are 3 eqn's with 6 variables, the IBFS is obtained by eqn $(6-3) = 3$ eqvariables to zero.

$\therefore$ The IBFS is $S_1 = 240$, $S_2 = 300$, $S_3 = 300$

$(x_1 = x_2 = x_3 = 0$, are non-basic$)$

Initial Simplex table:-

		C_j	2	2	4	0	0	0	Min Ratio
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_3
0	S_1	240	2	3	1	1	0	0	240
0	S_2	300	1	3	1	0	1	0	300
0	S_3	300	1	1	$\boxed{3}$	0	0	1	100 ←
$Z_j - C_j$		0	-2	-2	-4 ↑	0	0	0	

Entering variable $= x_3$

Departing variable $= S_3$

Pivot element $= 3$

$R_3(\text{new}) = R_3(\text{old}) \div 3$

$R_3(\text{old})$	300	1	1	3	0	0	1
$R_3(\text{new})$	100	1/3	1/3	1	0	0	1/3

$R_1(\text{new}) = R_1(\text{old}) - R_3(\text{new})$

$R_1(\text{old})$	240	2	3	1	1	0	0
$R_3(\text{new})$	100	1/3	1/3	1	0	0	1/3
$R_1(\text{new})$	140	5/3	8/3	0	1	0	-1/3

$R_2(\text{new}) = R_2(\text{old}) - R_3(\text{new})$

$R_2(\text{old})$	300	1	3	1	0	1	0
$R_3(\text{new})$	100	1/3	1/3	1	0	0	1/3
$R_2(\text{new})$	200	2/3	8/3	0	0	1	-1/3

First iteration :-

		C_j	2	2	4	0	0	0	Min Ratio $x_B/x_1, x_1>0$
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	140	5/3	8/3	0	1	0	-1/3	84 ←
0	S_2	200	2/3	8/3	0	0	1	-1/3	300
4	x_3	100	1/3	1/3	1	0	0	1/3	300
Z_j-C_j		400	-2/3 ↑	-2/3	0	0	0	4/3	

$Z_1 - C_1 = \frac{4}{3} - 2 = \frac{-2}{3}$

$Z_2 - C_2 = \frac{4}{3} - 2 = \frac{-2}{3}$

$Z_3 - C_3 = 4 - 4 = 0$

Entering variable $= x_1$

Departing variable $= S_1$

Pivot element $= 5/3$

$R_1(\text{new}) = R_1(\text{old}) \div (5/3) = R_1(\text{old}) \times 3/5$

$R_1(\text{old})$	140	5/3	8/3	0	1	0	-1/3
$R_1(\text{new})$	84	1	8/5	0	3/5	0	-1/5

$R_2(new) = R_2(old) - 2/3\, R_1(new)$

$R_2(old)$	200	2/3	8/3	0	0	1	-1/3
$R_1(new)$	84	1	8/5	0	3/5	0	-1/5
$\frac{2}{3} R_1(new)$	56	$\frac{2}{3}$	$\frac{16}{15}$	0	2/5	0	-2/15
$R_2(new)$	144	0	1.6	0	-2/5	1	-0.2

$R_3(new) = R_3(old) - 1/3\, R_1(new)$

$R_3(old)$	100	1/3	1/3	1	0	0	1/3
$R_1(new)$	84	1	8/5	0	3/5	0	-1/5
$\frac{1}{3} R_1(new)$	28	1/3	8/15	0	1/5	0	-1/15
$R_3(new)$	72	0	-0.2	1	-1/5	0	0.4

Second iteration :-

		C_j	2	2	4	0	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
2	x_1	84	1	1.6	0	0.6	0	-0.2
0	S_2	144	0	1.6	0	-2/5	1	-0.2
4	x_3	72	0	-0.2	1	-1/5	0	0.4
$Z_j - C_j$		456	0	0.4	0	0.4	0	1.2

$Z_1 - C_1 = 2 - 2 = 0$

$Z_2 - C_2 = 0.4$

$Z_3 - C_3 = 0$

$Z_4 - C_4 = 0.4$

$Z_5 - C_5 = 0$

$Z_6 - C_6 = 1.2$

Since all $(Z_j - C_j) \geq 0$ the current basic feasible solution is optimal.

$\therefore$ Max $Z = 456$

$x_1 = 84,\ x_2 = 0,\ x_3 = 72.$

5. Find the solution using Simplex method

$\text{Max } Z = 5x_1 + 10x_2 + 8x_3$

Subject to,

$3x_1 + 5x_2 + 2x_3 \le 60$

$4x_1 + 4x_2 + 4x_3 \le 72$

$2x_1 + 4x_2 + 5x_3 \le 100$

& $x_1, x_2, x_3 \ge 0$

Sol:- Problem is,

$\text{Max } Z = 5x_1 + 10x_2 + 8x_3$

Subject to,

$3x_1 + 5x_2 + 2x_3 \le 60$

$4x_1 + 4x_2 + 4x_3 \le 72$

$2x_1 + 4x_2 + 5x_3 \le 100$

& $x_1, x_2, x_3 \ge 0$

The problem is converted into standard form by adding slack variables.

$\text{Max } Z = 5x_1 + 10x_2 + 8x_3 + 0S_1 + 0S_2 + 0S_3$

ST, $3x_1 + 5x_2 + 2x_3 + S_1 = 60$

$4x_1 + 4x_2 + 4x_3 + S_2 = 72$

$2x_1 + 4x_2 + 5x_3 + S_3 = 100$

$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$

Initial Simplex table,

		C_j	5	10	8	0	0	0	Min ratio, $x_2 > 0$
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B / x_2
0	S_1	60	3	5	2	1	0	0	$60/5 = 12$ ←
0	S_2	72	4	4	4	0	1	0	$72/4 = 18$
0	S_3	100	2	4	5	0	0	1	$100/4 = 25$
	$Z_j - C_j$	0	-5	-10 ↑	-8	0	0	0	

Entering variable $= x_2$

Departing variable $= S_1$

Pivot element $= 5$

* $R_1(\text{new}) = R_1(\text{old}) \div 5$

$R_1(\text{old})$	60	3	5	2	1	0	0
$R_1(\text{new})$	12	3/5	1	2/5	1/5	0	0

* $R_2(\text{new}) = R_2(\text{old}) - 4R_1(\text{new})$

$R_2(\text{old})$	72	4	4	4	0	1	0
$R_1(\text{new})$	12	3/5	1	2/5	1/5	0	0
$4R_1(\text{new})$	48	12/5	4	8/5	4/5	0	0
$R_2(\text{new})$	24	8/5	0	12/5	-4/5	1	0

* $R_3(\text{new}) = R_3(\text{old}) - 4R_1(\text{new})$

$R_3(\text{old})$	100	2	4	5	0	0	1
$4R_1(\text{new})$	48	12/5	4	8/5	4/5	0	0
$R_3(\text{new})$	52	-2/5	0	17/5	-4/5	0	1

First iteration.

		C_j	5	10	8	0	0	0	Min Ratio, $x_3 > 0$
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_3
10	x_2	12	3/5	1	2/5	1/5	0	0	$\frac{12 \times 5}{2} = 30$
0	S_2	24	8/5	0	12/5	-4/5	1	0	$\frac{24 \times 5}{12} = 10$ ←
0	S_3	52	-2/5	0	17/5	-4/5	0	1	$\frac{52 \times 5}{17} = 60$
$Z_j - C_j$		120	-1	0	-4 ↑	2	0	0	

$Z_1 - C_1 = \frac{30}{5} - 5 = \frac{5}{5} = 1$

$Z_2 - C_2 = 10 - 10 = 0$

$Z_3 - C_3 = 4 - 8 = -4$

$Z_4 - C_4 = 2$

Entering variable $= x_3$

Departing variable $= S_2$

Pivot element $= \frac{12}{5}$

- $R_2(new) = R_2(old) \div (12/5) \Rightarrow R_2(old) \times \frac{5}{12}$

$R_2(old)$	24	8/5	0	12/5	-4/5	1	0
$R_2(new)$	10	2/3	0	1	-1/3	5/12	0

- $R_1(new) = R_1(old) - 2/5\, R_2(new)$

$R_1(old)$	12	3/5	1	2/5	1/5	0	0
$2/5\, R_2(new)$	4	4/15	0	2/5	-2/15	1/6	0
$R_1(new)$	8	1/3	1	0	1/3	-1/6	0

- $R_3(new) = R_3(old) - \frac{17}{5} R_2(new)$

$R_3(old)$	52	-2/5	0	17/5	-4/5	0	1
$\frac{17}{5} R_2(new)$	34	$\frac{34}{15}$	0	$\frac{17}{5}$	$-\frac{17}{15}$	$\frac{17}{12}$	0
$R_3(new)$	18	-8/3	0	0	1/3	$-\frac{17}{12}$	1

<u>Second iteration</u>

		C_j	5	10	8	0	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
10	x_2	8	1/3	1	0	1/3	-1/6	0
8	x_3	10	2/3	0	1	-1/3	5/12	0
0	S_3	18	-8/3	0	0	1/3	-17/12	1
$Z_j - C_j$		160	11/3	0	0	2/3	5/3	0

$Z_1 - C_1 = \left(\frac{10}{3} + \frac{16}{3}\right) - 5 = \frac{26}{3} - 5 = \frac{26-15}{3} = \frac{11}{3}$

$Z_2 - C_2 = 10 - 10 = 0$

$Z_3 - C_3 = 8 - 8 = 0$

$Z_4 - C_4 = \frac{10}{3} - \frac{8}{3} = \frac{2}{3}$

$Z_5 - C_5 = -10/6 + \frac{40}{12}$ (20/6)

$= \frac{10}{6} = \frac{5}{3}$

Since all $Z_j - C_j \geq 0$, the current basic solution is optimal.

$Max\ Z = 160$

$x_1 = 0,\ x_2 = 8,\ x_3\ 10$.

ARTIFICIAL VARIABLE TECHNIQUES

In these problems, atleast one of the constraints is of $=$ or $\geq$ type. To solve such LPP. There are two methods available.

i. The 'Big M-method' or 'Method of penalties'.

ii. The 'Two phase' method.

The Big M-method.

Step 1 :- Modify the constraints so that the RHS of each constraint is non-negative. After modification, identify each constraint as a $\leq, \geq$ or $=$ constraint.

Step 2 :- Convert each inequality constraint to standard form. (If a constraint is a $\leq$ constraint, then add a slack variable and if any constraint is a $\geq$ constraint, then subtract an excess variable, known as surplus variable).

Step 3 :- Add an artificial variable a_i to the constraints identified as '$\geq$' or with '$=$' constraints at the end of step 2. Also add the sign restriction $a_i \geq 0$.

Step 4 :- Let M denote a very large positive number. If the LP is a minimization problem, add $+Ma_i$ to the objective function. If LP is a maximization problem, add $-Ma_i$ to the objective function.

Step 5 : Solve the modified LPP by simplex method. While making iterations, using simplex method, one of the following three cases may arise.

i) If no artificial variable remains in the basis & the optimality condition is satisfied then the current solution is an optimal basic feasible solution

ii) If atleast one artificial variables appears in the basis at zero level & the optimality condition is satisfied, then the current solution is an optimal basic feasible Solution.

iii) At least 1 artificial variable appears in the basis at non-zero level & the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function since it contains a very large penalty M & is called pseudo optimal solution

ex:-

① Solve the following LPP by simplex method.

Maximize $Z = 3x_1 + 2x_2$

Subject to., $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

and $x_1, x_2 \geq 0$

Sol.- The problem is converted to canonical form by adding slack, surplus and artificial variables.

As the constraint-1 is of type '$\leq$' we should add slack variable S_1

As the constraint-2 is of type '$\geq$' we should subtract surplus variable S_2 & add artificial variable A_1

After introducing slack, surplus and artificial variables

$\text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$

Subject to, $2x_1 + x_2 + S_1 = 2$

$3x_1 + 4x_2 - S_2 + A_1 = 12$

and $x_1, x_2, S_1, S_2, A_1 \geq 0$

$S_1 = 2, A_1 = 12$ (basic) ($x_1 = x_2 = S_2 = 0$ non basic)

Iteration I.

		C_j	3	2	0	0	$-M$	Min Ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_2
0	S_1	2	2	1	1	0	0	$2/1 = 2$ ←
$-M$	A_1	12	3	4	0	-1	1	$12/4 = 3$
$Z_j - C_j$		$-12M$	$-3M-3$	$-4M-2$ ↑	0	M	0	

Entering variable = x_2

Departing variable = S_1

Pivot element = 1

$R_1(new) = R_1(old)$

$R_2(new) = R_2(old) - 4R_1(new)$

$R_2(old)$	12	3	4	0	-1	1
$4R_1(new)$	8	8	4	4	0	0
$R_2(new)$	4	-5	0	-4	-1	1

Iteration II.

		C_j	3	2	0	0	$-M$
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1
2	x_2	2	2	1	1	0	0
$-M$	A_1	4	-5	0	-4	-1	1
$Z_j - C_j$		$4-4M$	$5M+1$	0	$2+4M$	M	0

Since all $Z_j - C_j \geq 0$

Hence, optimal solution is arrived with value of variables as

$x_1 = 0,\ x_2 = 2$

Max $Z = 4$.

But this solution is not optimal & feasible because the solution violates the 2nd constraint $3x_1 + 4x_2 \geq 12$ & the artifical variable A_1 appears in the basis with positive value 4.

LPP posses "Pseudo optimal solution".

Q3 Use Big M-method to solve:-

Minimize $Z = 4x_1 + 3x_2$

Subject to, $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

and $x_1, x_2 \geq 0$

Sol:- $\text{Min } Z^* = -(\text{Max } Z)$

$\text{Max } Z^* = -4x_1 - 3x_2$

S.T., $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

& $x_1, x_2 \geq 0$

After introducing slack, surplus, artificial variables.

$\text{Max } Z = -4x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$

Subject To, $2x_1 + x_2 - S_1 + A_1 = 10$

$-3x_1 + 2x_2 + S_2 = 6$

$x_1 + x_2 + A_2 - S_3 = 6$

and $x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$

Iteration-1:

		C_j	-4	-3	0	0	0	$-M$	$-M$	Min Ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	X_B/x_1
$-M$	A_1	10	2	1	-1	0	0	1	0	$10/2=5$ ←
S_2	S_2	6	-3	2	0	1	0	0	0	-
$-M$	A_2	6	1	1	0	0	-1	0	1	$6/1=6$
$Z=-16M$		Z_j-C_j	$-3M+4$ ↑	$-2M+3$	M	0	M	0	0	

Entering $= x_1$, Departing $= A_1$, Key element $= 2$

$R_1(\text{new}) = R_1(\text{old}) \div 2$

$R_1(\text{old})$	10	2	1	-1	0	0	0
$R_1(\text{new})$	5	1	0.5	-0.5	0	0	0

$R_2(\text{new}) = R_2(\text{old}) + 3R_1(\text{new})$

$R_2(\text{new})$ 21 0 3.5 -1.5 1 0 0

$R_3(\text{new}) = R_3(\text{old}) - R_1(\text{new})$

$R_3(\text{new})$ 1 0 0.5 0.5 0 -1 1

Iteration-2

		C_j	-4	-3	0	0	0	$-M$	Min Ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	A_2	X_B/x_2
-4	x_1	5	1	0.5	-0.5	0	0	0	$5/0.5=10$
0	S_2	21	0	3.5	-1.5	1	0	0	$21/3.5=6$
$-M$	A_2	1	0	0.5	0.5	0	-1	1	$1/0.5=2$ ←
$Z=-M-20$		Z_j-C_j	0	$-0.5M+1$ ↑	$-0.5M+2$	0	M	0	

Entering $= x_2$, Departing $= A_2$, key element $= 0.5$.

Iteration-3

		C_j	-4	-3	0	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3
-4	x_1	4	1	0	-1	0	1
0	S_2	14	0	0	-5	1	7
-3	x_2	2	0	1	1	0	-2
$Z=-22$		Z_j-C_j	0	0	1	0	2

Since all $Z_j - C_j \geq 0$, the current solⁿ is optimal

$x_1 = 4$, $x_2 = 2$, Max $Z^* = -22$

$\therefore$ Min $Z = \underline{\underline{22}}$

Q4 Find solution using BigM method.

$\min Z = x_1 + x_2$

Subject to. $2x_1 + 4x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

and $x_1, x_2 \geq 0$

Sol:

$\text{Max } Z = -x_1 - x_2$

Subject to,

$2x_1 + 4x_2 \geq 4$

$x_1 + 7x_2 \geq 4$

and $x_1, x_2 \geq 0$

After introducing surplus, artificial variables.

$\text{Max } Z = -x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to,

$2x_1 + 4x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Iteration-1

		C_j	-1	-1	0	0	-M	-M	Min Ratio X_B/x_2
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	
-M	A_1	4	2	$\boxed{4}$	-1	0	1	0	$4/4 = 1$
-M	A_2	7	1	$\boxed{7}$	0	-1	0	1	$7/7 = 1$ ←
$Z = -11M$		$Z_j - C_j$	$-3M+1$	$-11M+1$ ↑	M	M	0	0	

Entering $= x_2$

Departing $= A_2$

Pivot / key element $= 7$.

- $R_2(\text{new}) = R_2(\text{old}) \div 7$

R_2(old)	7	1	7	0	-1	0
R_2(new)	1	0.1429	1	0	-0.1429	0

- $R_1(\text{new}) = R_1(\text{old}) - 4R_2(\text{new})$

R_1(new)	0	1.4286	0	-1	0.5714	1

Iteration 2.

		C_j	-1	-1	0	0	-M	Min ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_1
-M	A_1	0	(1.4286)	0	-1	0.5714	1	0 ←
-1	x_2	1	0.1429	1	0	-0.1429	0	7
Z = -1	$Z_j - C_j$		-1.4286M + 0.5571 ↑	0	M	-0.5714M + 0.1429	0	

Entering = x_1

Departing = A_1

Pivot element = 1.4286.

* $R_1(new) = R_1(old) \div 1.4286$
* $R_2(new) = R_2(old) - 0.1429 R_1(new)$

Iteration 3.

		C_j	-1	-1	0	0	Min ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	X_B/S_2
-1	x_1	0	1	0	-0.7	(0.4)	$\frac{0}{0.4} = 0$ ←
-1	x_2	1	0	1	0.1	-0.2	—
Z = -1	$Z_j - C_j$		0	0	0.6	-0.2 ↑	

Entering = S_2

Departing = x_1

Key element = 0.4.

* $R_1(new) = R_1(old) \div 0.4$
* $R_2(new) = R_2(old) + 0.2 R_1(new)$

Iteration-4

		C_j	-1	-1	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
0	S_2	0	2.5	0	-1.75	1
-1	x_2	1	0.5	1	-0.25	0
Z = -1,	$Z_j - C_j$		0.5	0	0.25	0

Since all $z_j - c_j \geq 0$

Hence optimal solution is arrived,

with $x_1 = 0$, $x_2 = 1$

$Max\ Z = -1$

$\therefore Min\ Z = 1$.

THE TWO PHASE METHOD.

Two-Phase Method Steps

Step-1 :- PHASE - 1

i. Form a new objective function by assigning zero to every original variable (including slack & surplus variables) and -1 to each of the artificial variables.

$Max\ Z = -A_1 - A_2$

ii. Using simplex method, try to eliminate the artificial variables from the basis.

iii. The solution at the end of Phase-1 is the initial basic feasible solution for phase-2.

Step-2 :- PHASE - 2

i. The original objective function is used and co-efficient of artificial variable is 0.

ii. Then simplex method is used to find optimal solution.

NOTE :- In step-1.

- If $Max\ z^* < 0$ and atleast one artificial variable appears in the optimum basis at a non-zero level. In this case the given LPP doesn't possess any feasible solution, stop the procedure.
- $Max\ z^* = 0$ and atleast one artificial variable appears in the optimum basis and zero level. In this case proceed to phase.

- Max $Z^* = 0$ & no artificial variable appears in the optimum basis. In this case proceed to phase-II.

Q1. Use two-phase simplex method to solve following LP problem.

Maximize $Z = -4x_1 - 3x_2 - 9x_3$

Subject to, $2x_1 + 4x_2 + 6x_3 \geq 15$

$6x_1 + x_2 + 6x_3 \geq 12$

$x_1, x_2, x_3 \geq 0$

Sol:- Convert general LPP into standard LPP.

$\leq$ - add a slack variable $(+S)$

$\geq$ - subtract a surplus variable, add an artificial variable $(-S+A)$

$=$ - add an artificial variable $(+A)$

Maximize $Z = -4x_1 - 3x_2 - 9x_3$

S.T., $2x_1 + 4x_2 + 6x_3 - S_1 + A_1 = 15$

$6x_1 + x_2 + 6x_3 - S_2 + A_2 = 12$

Find initial basic feasible soln,

$x_1 = x_2 = x_3 = S_1 = S_2 = 0$

$A_1 = 15,\ A_2 = 12$

<u>I-Phase</u> :- <u>Eliminate artificial variable</u>.

Assign a cost -1 to each artificial variable and cost 0 to other variable.

Auxiliary LPP, Max $Z^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1A_1 - 1A_2$

$2x_1 + 4x_2 + 6x_3 - S_1 + A_1 = 15$

$6x_1 + x_2 + 6x_3 - S_2 + A_2 = 12$

Construct the initial simplex table :-

I

C_B	B	C_j / X_B	0 / x_1	0 / x_2	0 / x_3	0 / S_1	0 / S_2	-1 / A_1	-1 / A_2	Min Ratio = x_B/entering variable
-1	A_1	15	2	4	6	-1	0	1	0	15/6 = 2.5
-1	A_2	12	6	1	6	0	-1	0	1	12/6 = 2 ←
		$Z_j - C_j$	-8	-5	-12 ↑	1	1	0	0	

$Z_j - C_j = \Sigma C_B x_j - C_j$

$Z_1 - C_1 = (-1\times 2) + (-1\times 6) - 0 = -8$

$Z_2 - C_2 = (-1\times 4) + (-1\times 1) - 0 = -5$

$Z_3 - C_3 = (-1\times 6) + (-1\times 6) - 0 = -12$

$Z_4 - C_4 = (-1\times(-1)) + (-1\times 0) - 0 = 1$

$Z_5 - C_5 = (-1\times 0) + (-1\times -1) - 0 = 1$

$Z_6 - C_6 = (-1\times 1) + (-1\times 0) - (-1) = 0$

$Z_7 - C_7 = (-1\times 0) + (-1\times 1) - (-1) = 0$

Entering variable = x_3

Departing variable = A_2

Pivot element = 6

$R_2(\text{new}) = R_2(\text{old}) \div$ key element (6)

R_2(old)	12	6	1	6	0	-1	0	1
R_2(new)	2	1	1/6	1	0	-1/6	0	1/6

$R_1(\text{new}) = R_1(\text{old}) - 6\,R_2(\text{new})$

R_1(old)	15	2	4	6	-1	0	1	0
R_2(new)	2	1	1/6	1	0	-1/6	0	1/6
$6R_2$(new)	12	6	1	6	0	-1	0	6
R_1(new)	3	-4	3	0	-1	1	1	-6

I

C_B	B	C_j / X_B	0 / x_1	0 / x_2	0 / x_3	0 / S_1	0 / S_2	-1 / A_1	-1 / A_2	Min ratio X_B/x_2
-1	A_1	3	-4	3	0	-1	1	1	-6	3/3 = 1 ←
0	x_3	2	1	1/6	1	0	-1/6	0	1/6	2×6 = 12
		$Z_j - C_j$	4	-3 ↑	0	1	-1	0	2	

Entering = x_2

Departing = A_1

Pivot element = 3.

III. $R_1(\text{new}) = R_1(\text{old}) \div 3$

$R_2(\text{new}) = R_2(\text{old}) - (1/6) R_1(\text{new})$

		C_j	0	0	0	0	0	-1	-1
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2
0	x_2	1	-4/3	1	0	-1/3	1/3	1/3	-1/3
0	x_3	11/6	11/9	0	1	+1/18	-2/9	-1/18	+2/9
		$Z_j - C_j$	0	0	0	0	0	1	1

All $(Z_j - C_j) \geq 0$

& $\text{Max } Z^* = C_B X_B = 0$

Phase II

Consider the final simplex table of phase I; also consider the actual cost associated with the original variables. Delete the artificial variables A_1, A_2 column from the table as these variables are eliminated from the basis in phase I.

		C_j	-4	-3	-9	0	0	Min ratio
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	$\theta = X_B/x_1$
-3	x_2	1	-4/3	1	0	-1/3	1/3	—
-9	x_3	11/6	11/9	0	1	1/18	-2/9	3/2 ←
	$Z_j - C_j$	$\frac{39}{2}$	-3 ↑	0	0	1/2	1	

Entering variable $= x_1$

Departing variable $= x_3$

Pivot element $= 11/9$

First iteration:-

		C_j	-4	-3	-9	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2
-3	x_2	3	0	1	3/2	-3/11	1/11
-4	x_1	3/2	1	0	9/11	1/22	$\frac{-2}{11}$
	$Z_j - C_j$	-15	0	0	$\frac{27}{11}$	7/11	5/11

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal

$\therefore$ The optimal solution is,

$\text{Max } Z = -15,$

$x_1 = 3/2,\ x_2 = 3,\ x_3 = 0$

Q. Use two phase simplex method to solve.

$\text{Maximize } Z = 5x_1 + 3x_2$

Subject to, $2x_1 + x_2 \leq 1$

$x_1 + 4x_2 \geq 6$

$x_1, x_2 \geq 0$

Sol: Phase 1

The problem is converted into standard form

$\text{Max } Z = 5x_1 + 3x_2 + 0S_1 + 0S_2$

S.T., $2x_1 + x_2 + S_1 = 1$

$x_1 + 4x_2 - S_2 + A_1 = 6$

and $x_1, x_2, S_1, S_2, A_1 \geq 0$

Phase 1 is to eliminate artificial variable. Assign a cost -1 to each artificial variable & cost 0 to other variable.

Auxilary LPP $\therefore$ $\text{Max } Z^* = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A_1$

$2x_1 + 3x_2 + S_1 = 1$

$x_1 + 4x_2 - S_2 + A_1 = 6$

& $x_1, x_2, S_1, S_2, A_1 \geq 0$

Construct initial simplex table,

Iteration I

		c_j	0	0	0	0	-1	Min Ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_2
0	S_1	1	2	$\boxed{1}$	1	0	0	$1/1 = 1 \leftarrow$
-1	A_1	6	1	4	0	-1	+1	$6/4 = 1.5$
	$z_j^* - c_j$		-1	$-4\uparrow$	0	1	0	

Entering variable $= x_2$

Departing variable $= S_1$

Pivot element $= 1$

$R_1(new) = R_1(old)$

$R_2(new) = R_2(old) - 4R_1(new)$

$R_2(old)$	6	1	4	0	-1	1
(−) $R_1(new)$	1	2	1	1	0	0
$4R_1(new)$	4	8	4	4	0	0
$R_2(new)$	2	-7	0	-4	-1	1

Iteration II

		c_j	0	0	0	0	-1
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1
0	x_2	1	2	1	1	0	0
-1	A_1	2	-7	0	-4	-1	1
	$z_j^* - c_j$		7	0	4	1	0

Since all $(z_j^* - c_j) \geq 0$, the current basic feasible solution is optimal.

$x_1 = 0, \; x_2 = 1$

Max $z^* < 0$ i.e. $z^* = -2$

But this soln is not feasible because

The solution violates the 2^{nd} constraint

$$x_1 + 4x_2 \geq 6.$$

and the artificial variable A_1 appears in the basis with positive value 2.

So, Phase-2 is not possible.

5. Using Two phase simplex method,

Minimize $Z = -2x_1 - x_2$

Subject to, $x_1 + x_2 \geq 2$

$x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

Sol:- Max $Z = 2x_1 + x_2$

ST, $x_1 + x_2 \geq 2$

$x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

Phase-1

Max $Z = -A_1$

ST,

$x_1 + x_2 - S_1 + A_1 = 2$

$x_1 + x_2 + S_2 = 4$

& $x_1, x_2, S_1, S_2, A_1 \geq 0$

Iteration 1:-

		C_j	0	0	0	0	-1	Min ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_1
-1	A_1	2	1	1	-1	0	1	$2/1 = 2$ ←
0	S_2	4	1	1	0	1	0	$4/1 = 4$

$Z = -2$

$Z_j - C_j$	-2	-1 ↑	-1	1	0	0

Entering variable $= x_1$

Departing $= A_1$ pivot element $= 1$

$R_1(new) = R_1(old)$

$R_2(new) = R_2(old) - R_1(new)$

Iteration - 2 ,

C_B	B	C_j / X_B	0 / x_1	0 / x_2	0 / S_1	0 / S_2
0	x_1	2	1	1	-1	0
0	S_2	2	0	0	1	1
	$Z_j - C_j$	0	0	0	0	0

Since all $Z_j - C_j \geq 0$

Hence, Optimal soln is arrived with value of variables as,

$x_1 = 2, x_2 = 0$

$Max\ Z = 0 \quad \therefore Min\ Z = 0$

Phase 2 :-

$Max\ Z = 2x_1 + x_2 + 0S_1 + 0S_2$

Iteration 1

C_B	B	C_j / X_B	2 / x_1	1 / x_2	0 / S_1	0 / S_2	Min ratio $X_B/S_1, S_1>0$
2	x_1	2	1	1	-1	0	—
0	S_2	2	0	0	1	1	2 ←
	$Z_j - C_j$	4	0	1	-2 ↑	0	

Entering variable $= S_1$

Departing variable $= S_2$

Pivot element $= 1$

$R_2(new) = R_2(old)$

$R_1(new) = R_1(old) + R_2(new)$

Iteration-2

		c_j	2	1	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
2	x_1	4	1	1	0	1
0	S_1	2	0	0	1	1
	$Z_j - C_j$	8	0	1	0	2

Since all $Z_j - C_j \geq 0$

Hence, the optimal solⁿ is obtained

$x_1 = 4, x_2 = 0$

Max $z = 8$ $\therefore$ Min $Z = -8$

Degeneracy (Tie for leaving basic variable)

During solving LP problem, a situation may arise in which there is a tie between, 2 or more basic variables for leaving the basis (means minimum ratios are same). It is called degeneracy.

To resolve this we can select any of them arbitrarily. But if artificial variable is present then it must be removed first.

Q. Find solution using Big M simplex method.

Max $Z = 3x_1 + 9x_2$

Subject to, $x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

and $x_1, x_2 \geq 0$

Sol:- The problem is converted to canonical form by adding slack, surplus & artificial variables as appropriate.

Max $Z = 3x_1 + 9x_2 + 0S_1 + 0S_2$

Subject to, $x_1 + 4x_2 + S_1 = 8$

$x_1 + 2x_2 + S_2 = 4$

and $x_1, x_2, S_1, S_2 \geq 0$

$(x_1 = x_2 = 0) \Rightarrow S_1 = 8, S_2 = 4$

1. Initial Simplex table.

		C_j	3	9	0	0	Min ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	$\theta = X_B/x_2$
0	S_1	8	1	$\boxed{4}$	1	0	$8/4 = 2 \leftarrow$
0	S_2	4	1	2	0	1	$4/2 = 2$
	$Z_j - C_j$	0	-3	$-9 \uparrow$	0	0	

Entering $= x_2$

Departing $= S_1$

Pivot Element $= 4$

$R_1(\text{new}) = R_1(\text{old}) \div 4$

$R_1(\text{new})$ 2 1/4 1 1/4 0

$R_2(\text{new}) = R_2(\text{old}) - 2R_1(\text{new})$

$R_2(\text{old})$	4	1	2	0	1
(−) $2R_1(\text{new})$	4	1/2	1	1/2	0
$R_2(\text{new})$	0	1/2	1	−1/2	1

<u>Iteration II</u>.

		C_j	3	9	0	0	Min ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	$\theta = X_B/x_1$
9	x_2	2	1/4	1	1/4	0	$2/1/4 = 8$
0	S_2	0	$\boxed{1/2}$	1	$-1/2$	1	$0 \leftarrow$
	$Z_j - C_j$	18	$-3/4 \uparrow$	0	9/4	0	

Entering $= x_1$

Departing $= S_2$ Pivot element $= 1/2$

$R_2(\text{new}) = \dfrac{R_2(\text{old})}{1/2}$ (or) $2\,R_2(\text{old})$

$R_2(\text{new})$	0	1	2	-1	2

$R_1(\text{new}) = R_1(\text{old}) - 1/4\, R_2(\text{new})$

$R_1(\text{old})$	2	1/4	1	1/4	0
$R_2(\text{new})$	0	1	2	-1	2
$\frac{1}{4}R_2(\text{new})$	0	1/4	1/2	-1/4	1/2
$R_1(\text{new})$	2	0	1/2	1/2	-1/2

III iteration,

		C_j	3	9	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
9	x_2	2	0	1/2	1/2	-1/2
3	x_1	0	1	2	-1	2
$Z_j - C_j$		18	0	3/2	3/2	3/2

Since all $Z_j - C_j \geq 0$

the current soln is optimal

$x_1 = 0$; $x_2 = 2$

Max $Z = 18$

Two-marks questions.

1. Define the term LPP.

A:- Linear programming is a method for maximizing or minimizing a linear function of several variables, subject to a set of linear constraints.

2. What are the applications of Linear programming.

A:- LP has a wide range of applications across various industries & fields. Some common applications include.

(i) Supply chain management. (iii) Agricultural planning
(ii) Production planning (iv) Resource allocation
(v) Transportation & Logistics.

3. What is simplex method?

A:- Simplex method is most powerful method, it deals with iterative process, which consists of first designing a Basic feasible solution or a programme and proceed towards the Optimal solution & testing each feasible solution for optimality to know whether the solution is optimal or not.

4. Define slack variable and surplus variable.

A:- Slack variable:- Slack variable represents an unused quantity of resources; it is added to '≤' type constraints in order to get an equality constraint

Surplus variable:- Surplus variable represents the amount by which solution values exceed a resource. These variables are also called 'negative slack variab[les]'. It is added to '≥' type constraints in order to get an equality constraint.

6. What is a Basic variable?

A. Basic variables are the variables with co-efficients '1' in the equations and '0' in the other equations.

. Define Artificial variable

A. Artificial variables are fictitious & cannot have any physical meaning. These are introduced in LP to convert inequality constraints into equality constraints.

7. Define Pivot element.

A. In simplex method, a pivot element is the entry where the pivot row & the pivot column meet in the simplex table. The pivot element is used to perform a pivot operation that replaces a basic variable with a non-basic variable.

8. What is Big M-method?

A. The Big-M method is another method of removing artificial variables from the basis. In this method, we assign co-efficients to artificial variables, undesirable from the objective function. If objective function Z is to be minimized, then a very large positive price (called penalty) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called penalty) is assigned to each of these variables. The penalty will designated by -M for a maximization problem and +M for a minimization problem, where $M > 0$.

Q. What is operation research?

A. Operations research is the systematic application of quantitative methods, techniques & tools to the analysis of problem involving the operations of systems.

Q. Write some applications of OR.

A:- ① Manufacturing :-

- Inventory control
- Marketing balance projection.
- Production scheduling.

② Marketing :-

- Advertising budget allocation
- Product introduction timing.
- Selection of product mix.

③ Purchasing :-

- Optimal buying
- Optimal reordering
- Materials transfer.

④ Construction

- Project scheduling, monitoring & control
- Determination of proper work force
- Allocation of resources to projects